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A HEURISTIC APPROACH TO PROBLEM SOLVING IN GRADE ELEVEN MATHEMATICS

by

© DEAN FREDERICK TWEEDLE

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled A Heuristic Approach to Problem Solving in Grade Eleven Mathematics submitted by Dean Frederick Tweedle in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

The fundamental purpose of this thesis was to develop a unit for training students in a heuristic problem-solving process. The major thrust of this unit was to instruct pupils in the transfer process. That is, students were trained to characterize a problem, to associate characteristics with a method of solution, and to use this solution method to solve the problem. The problem under consideration in this research study was to develop the above-mentioned unit and to determine whether the unit, when taught to grade eleven (Mathematics 20) students, yielded any significant difference in problem-solving test scores when compared to a more conventional approach to problem solving.

Three essential steps comprised this investigation:

(1) The Development of the Unit. The author developed an instructional strategy, the Heuristic Problem-Solving Process (HP), for the implementation of Polya's heuristic approach to problem solving. This strategy had as its framework, the general steps of the heuristic process: understanding the problem, devising a plan, carrying out the plan, and looking back. Within this framework, the students were taught five methods of solution: generalization, specialization, analogy, decomposing and recombining, and working backward.

(2) The Development of a Control Treatment. The conventional problem-solving process (CP) was taught to the control group. The same problems were used with the control group as were used with the experimental group, but the major thrust of HP was excluded.

(3) The Structure of the Treatment. Two grade eleven (Mathematics 20) classes in the Edmonton Public School System participated in the study. The author taught the experimental (N = 27) and control (N = 32) groups respectively through HP and CP for four weeks. One pretest and two post-tests were administered to each group as well as a Student Opinion Survey (SOS).

Complete data was obtained for thirty-six students (experimental N = 19; control N = 17). The pretest was utilized as the equating instrument. Classroom observations, and the comments on the SOS, and the Cooperating Teacher Opinion Survey (COS) confirmed satisfactory implementation of HP and CP. Two-way analyses of variance of the post-test measures showed that: (1) HP and CP did not differ significantly on a problem-solving test containing problems directly related to the treatments; (2) HP and CP did not differ significantly on a problem-solving test containing problems typical to the Mathematics 20 curriculum. A one-way analysis of variance, using the experimental group, showed a significant improvement ($\alpha = .01$) from pretest to post-test. Although the control group made a similar improvement, the SOS and COS revealed a much more positive attitude toward the HP.

The major conclusion of this study is that a problem-solving unit, either HP or CP, will produce a marked improvement in the students' problem-solving scores.

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It is certainly a fact that a wife makes an invaluable contribution, in providing an environment that is conducive to concentrated study and long hours of unbroken silence. It is, therefore, in deepest gratitude that I dedicate this thesis to her.

Finally, I feel constrained to make mention of One to whom I am forever grateful; that is, my Creator and Savior, the Lord Jesus Christ. He gives me the peace of mind, the guidance, the love, and the strength, as I wait on Him, for which I will always be thankful.

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CHAPTER I

INTRODUCTION TO THE PROBLEM

A. Background to the Problem

Stilwell (1967) reported in his research study that only 3% of the total time spent on problem-solving activity was devoted to developing a method for solving problems. This statistic is misleading in the sense that the mere act of attempting to solve problems involves developing some method or strategy that enables the problem solver to progress through the problem and hopefully arrive at the correct solution. However, the statistic does emphasize a need to study the methods for solving problems.

It is in response to this need, to study the methods for solving problems, that the author wished to focus his attention. In particular, the author sought to respond through the development of a teaching strategy for training students in a heuristic problem-solving process. There have been other researchers who shared a similar concern and their research studies are a partial fulfillment of this need. The present picture is incomplete though: some studies reported significant results, thus reinforcing the need for students to study the methods of solving problems; other studies reported no significant differences. Crutchfield (1965: 13-15) reported on some highly successful and significant work he had done in teaching children (through self-instructional programmed booklets) to apply heuristic strategies to non-mathematical problems. Larsen (1960) taught a group

of university students to apply heuristic strategies to mathematical problems. Ashton (1962) found that ninth-grade algebra students, taught by the heuristic method, showed greater improvement in problem solving (as measured by achievement test scores), than those students who were taught by the textbook method. Kilpatrick (1967) developed a technique for analyzing behavior during the solution of a mathematical problem, and through this technique found that few students attempted to derive a solution by more than one method.

Wilson (1967), Kilpatrick (1969), and Shulman (1970) in their studies, noted the dearth of research in the area of the development and analysis of processes involved in problem solving. Henderson and Pingry (1953) and Gurova (1969) stated that practice in solving problems and a conscious awareness of the problem-solving processes will improve problem-solving performance.

Lucas (1974: 36-46) conducted an exploratory study for the purpose of analyzing the influence of heuristic-oriented teaching on a group of first-year university calculus students. He found that college students who received training in heuristic processes exhibited superior overall performance in approaching a problem devising workable plans, and obtaining accurate results. Bassler, Beers, and Richardson (1975: 170-177) conducted a study in which they compared two strategies for instructing students in problem solving: one strategy was derived from the work of Polya and was basically heuristic in nature; the other one involved a direct translation of the verbal statement into mathematical symbolism. Their results showed that there was no difference between the means

of these groups on the problem-solution criteria. Post and Brennan (1976) investigated the effects of various modes of presentation of general heuristic processes on tenth-grade students' problem-solving ability. Their results revealed students who received instruction in the General Heuristic Problem-Solving Procedure obtained no significantly better results on the problem-solving test utilized in their study than students who received no such training. Kantowski's (1977) clinical, exploratory study described the changes in the problem-solving processes of eight above-average ninth-grade students as they learned to solve nonroutine geometry problems. She noted in her results that "looking back" strategies did not increase with the students' problem-solving ability nor was it, apparently, related to success in problem solving.

In connection with the previously mentioned studies, the following points should be noted:

1. Crutchfield's study dealt with non-mathematical problems and he utilized a programmed-instruction method of teaching.
2. Larsen's and Lucas' studies were conducted on samples of university students.
3. Ashton's results are limited by the content of her problems; that is, although she taught her students the mathematical problem-solving checklist and how to ask the appropriate question at the appropriate time, her choice of problems required that the students only have to know one method of solution.

4. Kilpatrick's (1967) study did not develop a heuristic problem-solving strategy but developed a technique for analyzing behavior during the solution of a mathematical problem. One result of his analysis was that few students attempted to derive a solution by another method. This result further magnifies the limitation in Ashton's study.
5. Wilson's study was conducted with high school students utilizing self-instructional booklets. One implication of his study was that general heuristics (that is, those not bound to specific tasks), should be incorporated as part of problem-solving training in the classroom.
6. Kilpatrick (1969), in his paper, noted the dearth of published studies on problem solving at the secondary school level.
7. Shulman, in his paper, noted a proper psychological base cannot be given to mathematics instruction because mathematicians view mathematics from two different perspectives, either as process learning or as subject matter understanding.
8. Henderson and Pingry, in their paper, asserted that there is only one way in which students could learn problem solving—by solving problems and studying the processes.
9. Gurova's study of twenty-one fifth and sixth-grade pupils revealed that a relevant external stimulus could

help a student become conscious of the mental processes that he used in solving a problem.

10. Bassler, Beers and Richardson's study has results which are of limited applicability to the heuristic problem-solving process. Although their first strategy was heuristic in nature and derived from the work of Polya, it did not use the mathematical problem-solving checklist advocated by Polya and it also carried the same limitation as previously mentioned with regard to Ashton's study.
11. Post's and Brennan's study utilized tenth-grade students who were studying geometry. Their results revealed that students who received instruction in the General Heuristic Problem-Solving Procedure obtained no significantly better scores on the problem-solving test used in their study, than students who received no such training.
12. Kantowski's study was a clinical investigation conducted on eight above-average ninth-grade students who were learning to solve nonroutine geometry problems. She reported that the "looking back" strategies did not appear to be related to success in problem solving.

Ong's (1976) study provided the incentive for the author's thesis. His study revealed that a creative problem-solving model could be operationalized through the development and implementation of his instructional method. According to Ong (1976: v,vi):

> The major conclusion of the study is that through appropriate operationalization of Boychuk's model of creative problem solving, an instructional method for enhancing student mathematical creativity could be developed and successfully implemented in a significant content domain of junior high school mathematics curriculum.

In this thesis, the author has developed and implemented a heuristic problem-solving unit of instruction based on Polya's model at the grade eleven level. Unlike either Crutchfield's (1967) or Wilson's (1967) self-instructional mode, the author's treatment utilized direct teacher involvement. The author's study was not a clinical investigation as was Kantowski's nor did it confine itself to a specific subject matter domain.

Progressing from the limitations of Ashton's (1962) study and based upon the recommendations of Kilpatrick (1967, 1969), Henderson and Pingry (1953), Wilson (1967), and Gurova (1969) the author developed a general heuristic framework for solving problems; incorporating within this approach, five methods of solution. This instructional mode is contained within a problem-solving unit which has been designed to be taught as a separate unit in the Mathematics 20 curriculum. Henderson and Pingry (1953), Gurova (1969), and Kantowski (1977) placed a heavy emphasis on the teaching of process in problem solving; in response to their convictions, the author has developed a unit for training students in a heuristic problem-solving process.

B. Statement of the Problem

The fundamental purpose of this thesis is to develop a unit for training students in a heuristic problem-solving process. The major

thrust of this unit is to instruct pupils in the transfer process. That is, students were trained to characterize a problem, to associate the characteristics with a method of solution, and to use this solution method to solve the problem. Since the work of Polya is most noteworthy on this subject, the author developed an instructional strategy for the implementation of Polya's heuristic approach to problem solving.

The problem under consideration in this research study was to develop the above-mentioned unit and to determine whether the unit, when taught to grade eleven (Mathematics 20) students, yielded any significant difference in problem-solving test scores when compared to a more conventional approach to problem solving. This investigation seeks answers to the following questions:

1. What attitudes do students and teacher have toward a heuristic problem solving <u>instructional strategy</u>?
2. Does such an instructional <u>strategy</u> affect the students' ability in solving those problems which are peculiar to those utilized in that <u>strategy</u>?
3. Does such an instructional strategy affect the students' ability in solving those problems which are peculiar to those utilized at a specific <u>grade</u> and <u>course level</u>?

C. Definition of Terms and Variables

1. Heuristic

In this study, Polya's definition of heuristic was utilized.

> I wish to call heuristic, the study that the present work attempts, the study of the means and methods of problem solving (Polya, 1962: vi).

> Heuristic reasoning is reasoning, not regarded as final and strict, but as provisional and plausible only, whose purpose it is to discover the solution of the present problem (Polya, 1957: 113).

> Finished mathematics presented in a finished form appears as purely demonstrative, consisting of proofs only. Yet mathematics in the making resembles any other human knowledge in the making . . . . The result of the mathematician's creative work is demonstrative reasoning, a proof; but the proof is discovered by plausible reasoning, by guessing . . . . In plausible reasoning, the principle thing is to distinguish a guess from a guess, a more reasonable guess from a less reasonable guess (Polya, 1954: vi).

2. Heuristic Approach to Problem Solving

This approach is the experimental treatment of this research study. It is delineated in Chapter IV of this thesis, entitled "The Heuristic Problem-Solving Method of Instruction."

3. Conventional Approach to Problem Solving

This approach is the control treatment of this research study. It is delineated in Chapter III, section A, subsection 5 (b), entitled "Control Treatment."

4. Transfer of Training

Information and habits that one has built up in the past will affect new learning. According to Ellis (1965: 3):

> Transfer of training means that experience or performance on one task influences performance on some subsequent task.

In the experimental treatment of this research study, the problem solver was trained to characterize a problem, to associate the characteristics with a method of solution, and to use this solution method to solve the problem. Subsequently, the problem solver was tested by being presented with problems whose solutions were unknown. He was expected to characterize those problems, to choose the appropriate methods of solution, and to use them to solve the problems. The influence of the tasks learned in the training phase, upon the problems presented in the test phase is referred to as transfer of training.

5. Problem

Although Thorndike's definition of a problem was given in the context of his animal experimention, it nevertheless is still widely accepted in the context of human problem solving.

> A problem exists when the goal that is sought is not directly attainable by the performance of a simple act available in the animal's repertory; the solution calls for either a novel action, or a new integration of available actions (Scheerer, 1963: 118).

Polya (1962: 117), in defining a problem, reiterated Thorndike's earlier definition:

> Thus to have a problem means: to search consciously for some action appropriate to attain a clearly conceived, but not immediately attainable aim.

D. Delimitations

The researcher has chosen to delimit the study in several areas. The sample was restricted to students within the Edmonton Public School System enrolled in Mathematics 20. Another area of delimitation concerns the type of problems contained in the study. Only those

problems, which can be solved by the five methods of solution presented in this study, were chosen. Also, the geometry and algebra content contained in the mathematics word problems represents only a small sample of possibilities.

E. Limitations

In the selection of the experimental group and the control group within the same school it is assumed that the previous mathematical experiences of the students in each class are similar. Another limitation is the assumption that the two classes will adequately represent all the Mathematics 20 students in the Edmonton Public School System.

CHAPTER II

LITERATURE REVIEW AND CONCEPTUAL FRAMEWORK

A. Research Studies on the Heuristic Process

There has been only a limited number of studies done in the area of heuristic problem solving. Crutchfield (1965: 13-15) reported on some highly successful and significant work he had done in teaching children to apply heuristic strategies to non-mathematical problems. He used self-instructional programmed booklets with semi-cartoon format to teach creative thinking and problem-solving skills to fifth and sixth grade pupils. The booklets present the continuing story of a brother and sister, Jim and Lila, as they try to solve a series of puzzles and mysteries with the aid of their uncle John, a high-school science teacher and part-time detective. The pupils are supposed to identify with Jim and Lila as they gradually learn to overcome their anxieties and become proficient in problem solving. The booklets not only give the pupil repeated experiences in the solution of interesting problems, but they also show him an array of strategies: formulating the problem, asking relevant questions, planning one's attack, generating many ideas, searching for uncommon ideas, tranforming the problem, evaluating hypotheses, sensing discrepancies, and using analogies. Crutchfield has had remarkable success in improving attitudes toward problem solving and in getting children to transfer skills learned in the training program to problems of quite a different type.

Larsen (1960) taught a group of university students to apply heuristic strategies to mathematical problems. He showed that calculus could be taught to illustrate heuristic notions, but he failed to obtain improved problem-solving performance.

Ashton (1962), in her study, addressed the problem: Would ninth-grade students taught by the heuristic method show greater improvement in problem solving (as measured by achievement test scores) than those students who were taught by the textbook method? The experimental group were taught to ask key questions such as: What is the unknown? What are the data? What are the conditions? These questions (from Polya) were designed to help the students progress through the solution. The control group using the textbook method for solving problems was given problems similar to those given to the experimental group. For the control group the procedure for solving problems of a particular type was demonstrated for the pupils who were then assigned similar problems for practice. Ashton found that after ten weeks of instruction, ninth-grade students who were taught to use the heuristic method for solving problems were better able to solve verbal algebra problems than the students who were taught the conventional textbook method.

Kilpatrick (1967: 46) shared a similar concern with Ashton: that is, how students solve problems. Ashton wanted to compare methods of problem solving, whereas Kilpatrick attempted to develop a technique for analyzing behavior during the solution of a mathematical problem and to demonstrate how this technique could be used to delineate an individual's approach to a broad class of problems. Fifty-six

subjects of both sexes and above average mental ability, who had just completed the eighth grade were interviewed individually and asked to think aloud as they solved a battery of mathematical problems. As a result of the pilot study, Kilpatrick chose eleven strategies from the heuristic process. These eleven strategies were chosen because they were relatively easy to code, they occurred more frequently than the other heuristics, and they were of particular interest to the investigator. The modified list is shown below:

Understanding the Problem
1. Identifies unknown, data, or condition
2. Draws figure
3. Introduces notation

Devising a Plan
1. Rephrases the problem
2. Considers a related problem (special case of part of the problem)

Carrying Out the Plan
1. Uses successive approximations
2. Checks steps before final result

Looking Back
1. Checks that result is reasonable
2. Checks that result satisfies conditions
3. Retraces steps of argument
4. Derives result by another method

When the coding system was applied to the pupils' tape-recorded protocols, only a few of the strategies described by Polya were observed. Few students, for instance, varied the conditions of the problem, recalled having solved similar problems, or attempted to derive a solution by another method. Although most subjects drew figures while solving the problems, the frequency with which figures were drawn was unrelated to success in solving problems or to the other strategies. The use of trial-and-error was widespread, but its

disciplined use, as successive approximation, was much less frequently found.

Wilson (1967), Kilpatrick (1969), and Shulman (1970) in their studies noted the dearth of research in the area of the development and analysis of processes involved in problem solving. Wilson's (1967) study "investigated the problem-solving performance of high school students on learning tasks and transfer tasks following instruction in which the level of generality of the problem-solving heuristics taught was an independent variable." The students were trained, using self-instructional booklets, one group to each of the three levels of generality in problem-solving heuristics: task specific heuristics, means-ends heuristics, and planning. The evidence from his study indicates that: "General heuristics such as means-ends heuristics or planning heuristics facilitate transfer and should be incorporated as part of problem-solving training in the classroom."

Kilpatrick (1969), in his literature review on problem solving and creative behavior mentioned the prevalence of studies done in this area in elementary school mathematics but the dearth of studies conducted in the higher grades. He said (1969:154):

> Similar studies at the secondary school level would be useful, if only to document the dearth of published studies on problem solving at the higher grades.

Shulman (1970), in his investigation of research studies conducted in the areas of psychology and mathematics education, concluded that no single psychological theory would satisfy all mathematics educators seeking to develop mathematics instruction. The reason was the two different perceptions mathematics researchers have of mathematics.

According to Shulman (1970: 70):

> . . . if one perceives "mathematics" as basically a body of strategies, heuristics, or methods of inquiry, then clearly an approach to instruction calculated to optimize process learning is most advisable. If mathematics is seen as a compendium of subject matter understanding (e.g., arithmetic facts, computational algorithms, specific postulates or theorems), an approach which optimizes subject matter mastery would be preferred.

Bruner (1966: 72) stated his preference for the two perceptions of mathematics noted by Shulman when he said:

> We teach a subject not to produce little living libraries on that subject, but rather to get a student to think mathematically for himself, to consider matters as a historian does, to take part in the process of knowledge-getting. Knowing is a process, not a product.

Henderson and Pingry (1953) and Gurova (1969) stated that practice in solving problems and a conscious awareness of the problem-solving processes will improve problem-solving performance. Henderson and Pingry (1953) made a number of assertions in their study based upon the findings of previous researchers. In answer to the question: Should mathematics courses contain more problems?, they replied:

> For what we know about learning, there is only one way students can learn to solve problems—by solving problems and studying the process (1953: 233).

> Unless students study the process of solving problems as an end in itself there is scant likelihood that they will learn the generalizations which will enable them to transfer their ability to solve problems to new problems as they arise (1953: 233).

> Mathematics teachers need to be students of problem-solving processes as well as students of mathematics. There is considerable evidence that many mathematics teachers do not understand what problem solving is; or if they know, they do not have it as an objective of instruction (1953: 249).

Gurova (1969) investigated the process of solving arithmetic problems of twenty-one fifth and sixth-grade pupils. He discovered that students whose attention was directed to a problem's conditions and solution method could correct calculation errors (in most cases) and became aware of their reasoning in solving the problem. According to Gurova (1969: 101):

> The experiments showed that if a relevant external stimulus turns a student's thinking toward analysis of a problem's conditions and the way to solve it, the student not only corrects factual errors in or advances his solution, but also becomes aware of the reasoning behind both correct and incorrect operations—that is, he begins to solve the problem consciously.

Lucas (1974: 36-46) conducted an exploratory study for the purpose of investigating heuristic usage and problem-solving performance, and analyzing the influence of heuristic-oriented teaching on a group of first-year university calculus students. One result of Kilpatrick's study was that few students attempted to derive a solution by another method. Lucas used tasks that were unstructured word problems involving a free choice of alternatives. Thirty students were divided into two groups, an experimental group (17 students) and a control group (13 students). The control group was given only expository treatment of problem solutions with minimal attention to heuristic strategies. In contrast, the experimental group had the inquiry technique applied during the discussion of problems, where the objects of inquiry were heuristic strategies. Lucas utilized a clinical approach in his research study in which he employed a modification of the system of behavioral analysis originally devised by Kilpatrick. His results were:

1. that the experimental group utilized mnemonic notation significantly more than the control group;
2. that the frequency of diagrams showed no correlation to achievement score (result supports Kilpatrick's finding);
3. that using the method or the result of a related problem (two heuristic strategies) to assist in producing a solution to the problem at hand, was affected by heuristic instruction;
4. that there were no significant differences between the experimental and the control groups with respect to the number or kind of errors committed during the process of solving a problem;
5. that college students who received training in heuristic process exhibited superior overall performance in approaching problems, devising workable plans, and obtaining accurate results.

It should be remembered, when reading the results of Lucus' study, that his sample size is small (13 control pupils and 17 experimental pupils), and therefore his results could be suspect.

Kantowski (1974) conducted a longitudinal study in which she observed the problem-solving behaviors of eight grade ten students as they solved geometry problems. The students were given instruction in two phases, the first stressed problem-solving strategies rather than content and the second phase introduced geometry content. A clinical, "think aloud" methodology was employed. She concluded that: if the heuristics used were goal oriented, the solution tended to be more efficient; failure to introduce a heuristic often led to many superfluous analyses; and generally, the tendency to use heuristics increased as problem-solving ability developed.

Bassler, Beers, and Richardson (1975: 170-177) conducted a study in which they compared two strategies for instructing students in problem solving: one strategy was derived from the work of Polya and was meant to be heuristic in nature; the other strategy involved a direct translation of the verbal statement into mathematical symbolism.

The results showed that there was no difference between the means of these groups on the problem-solution criteria. Although this study purports to use a heuristic strategy as one of its two problem-solving approaches, the steps (listed below) are more algorithmic in composition and therefore allow almost no room for an individual to employ plausible, inductive reasoning (characteristics of Polya's heuristic approach). The six steps are:

1. Read the problem carefully.
2. Decide what question the problem asks and choose a variable to represent the unknown.
3. Consider the other information given in the problem and how it relates to the unknown.
4. Write an equation or equations expressing the given relationships.
5. Solve the equation or equations.
6. Check the answer.

There is a basic difference between algorithms and heuristic. Algorithms are a series of sequential rigid steps that require little (if any) exploratory thinking on the part of the problem solver. According to Scandura (1966: 1-6), algorithms can be applied mechanically without the problem solver having to understand what he is doing and why he is doing it. Heuristic, on the other hand, is a more flexible, open-ended approach to problem solving. It operates through a process of discovery, by means of plausible, inductive reasoning. It is therefore the opinion of the author that the results of this study should not be attributed to a heuristic approach.

Webb (1975) observed the problem-solving processes of forty grade XI students as they solved mathematical problems while thinking aloud. A modification of the Kilpatrick coding was used to code the students' responses. He concluded that: (1) mathematics achievement had the

highest relation to problem-solving ability; (2) the use of heuristic strategies did account for some of the variance (13%) in problem-solving ability; (3) most of the heuristic strategies studied were problem-specific; (4) those students who used a range of these strategies were on the average better problem solvers.

Grady (1975) conducted a study, utilizing thirty-three grade IX students, in which he examined the relationship between mathematical word problem-solving performance and level of operational thought in Piagetian terms. He administered a Piagetian task instrument, a general reasoning test and two problem-solving tests. An analysis of the results indicated that: concrete and formal operational students did not differ in their frequencies of use of planning heuristics but formal operational subjects used more means-end heuristics than did concrete operational subjects; successful problem solvers used significantly more heuristics than did unsuccessful problem solvers; there was no difference in the use of deductive reasoning and trial-and-error between successful and unsuccessful problem solvers.

Pereira-Mendoza's (1975) study investigated the effect of teaching heuristics on the ability of 189 grade X students to solve novel mathematical problems. He used three treatments in his study: the heuristics only group; the combination of heuristics and content group; and the content only group. A second factor concerned employing distinct instructional materials for the treatments, namely algebraic, geometric, and mathematically neutral materials. From his study he concluded that: there were no overall significant differences involving the three kinds of instructional materials utilized; the heuristic-only

group scored significantly better on the algebraic test; both the heuristic-only group and the heuristic-with-content group scored better than the content-only group on the geometry test.

Post and Brennan (1976) investigated the effects of various modes of presentation of general heuristic processes on the students' problem-solving ability. The experimental treatment group was provided with a teacher-directed, group-paced learning package where the emphasis was placed on solving problems using the General Heuristic Problem-Solving Procedure (GHPSP). The control group received no formal instruction in problem solving but merely continued on with the normal instruction provided in tenth-grade geometry. The GHPSP was operationally defined as follows:

I. Recognition, Clarification, and Understanding of the Problem
   1. Read the problem carefully.
   2. Look up any words you do not understand.
   3. What is the unknown? What are the data? What is the condition? What is given?
   4. State the problem in your own words.
   5. Break the problem into parts.
   6. Draw a diagram to aid in clarification.
   7. Accept or reject the specific problem tools as a problem for yourself.

II. Plan of Attack—Analysis
   1. Gather data (facts, rules, relationships) which are necessary for solution.
   2. Recall missing data, select relevant data from the problem statement, and generate new data if necessary.
   3. Eliminate irrelevant data.
   4. Decide on needed approach activities by noting obstacles to the solution of the problem.

III. Productive Phase
   1. Find the connection between the data and the unknown. You may have to consider auxiliary problems if an immediate connection cannot be found. Do you know a related problem?
   2. Generate a hypothesis or a number of alternative hypotheses (possible solutions of the problem).

3. Order your data in preparation for hypothesis testing.
4. Reject initial hypotheses that do not satisfy the conditions of the problem.
5. Select a remaining hypothesis for testing.
6. Construct an algorithm or develop a heuristic for the manipulation of data as an instrument for possible verification of a hypothesis.

IV. Validating Phase—Checking—Proving
1. Accept or reject the hypothesis by verifying or not verifying that it meets the conditions of the task.
2. Look back. Can you check the result? Can you check the argument? Can you derive the result differently? Can you use the result or the method for some other problem?
3. If you have rejected your hypothesis, select a remaining one for testing. (Post and Brennan, 1976: 59,60)

The results revealed that students who received instruction in the GHPSP obtained no significantly better results on the problem-solving test used in their study.

Vos (1976) in his study examined the effects of three instructional strategies on problem-solving behaviors in 133 secondary school mathematics students. His sample was taken from the grade IX, X and XI classes in a private school. The instructional strategy of each of the three treatment groups was as follows: one group received the problem without any instruction; another group received the problem followed by the instruction; and the third group received instruction followed by the problem. The results indicated that the students given instruction on specific behaviors were more likely to exhibit and make effective use of the problem-solving behaviors in new problem situations.

Kantowski's (1977) clinical, exploratory study described the changes in the problem-solving process of eight above-average ninth-grade students as they learned to solve nonroutine geometry problems.

Initially, she differentiated between the two aspects of problem solving, process and product.

> Similarly, problem solving has two aspects: (a) the process, or set of behaviors or activities that direct the search for the solution, and (b) the product, or the actual solution. Both the process and the product are essential components of the problem-solving experience (Kantowski, 1977: 163).

In reporting her results, Kantowski (1977: 169) mentioned that:

> . . . the "looking back" strategies did not increase as problem-solving ability developed, nor did it appear to be related to success in problem solving.

Another area in which research studies on the heuristic process have of late been concentrated, is the field of computer simulation of the human problem-solving processes; that is, in the development of computer problem-solving models. Kilpatrick (1969) substantiated the importance of this area of research when he related it to Polya's work in heuristic problem solving. He said (1969: 163):

> Modern interest in heuristics—the study of the methods and rules of discovery and invention—is due principally to Polya (1957, 1962, 1965), who set forth maxims for problem solving which, he postulates, correspond to mental actions. Evidence for the validity of Polya's observations on the problem-solving process has come most strikingly from work on computer simulation of human behavior.

Models are analogies of real phenomena; they are isomorphic to the real object or process. According to Davis (1973: 60): ". . . the more accurately the details of the model correspond to and predict the details of the real object or process, the better the model." One model, of growing importance today, is the computer; a simulation of complex human behavior.

The term 'artificial intelligence' encompasses all computer

programs that play games (for example, cards and chess), retrieve facts, recognize figural patterns, do mathematics (algebra, geometry, and integral calculus), learn concepts and paired-associate lists, make decisions based upon various combinations of reward probabilities, and complete number of letter sequences. However, within artificial intelligence are two simulated problem-solving techniques: algorithmic programs and heuristic programs. The algorithmic program will produce humanlike solutions without necessarily following a human-like sequence of operations. The computer will explore all possible solution-combinations in some predetermined order. The heuristic program, on the other hand, will proceed through the search process and solve the problem in roughly the same manner as a human being. According to Davis (1973, 62, 74):

> The forte of the heuristic program is its ability to drastically reduce the amount of mechanical trial-and-error searching by selectively pursuing only goal paths that appear promising. Unlike the powerful algorithmic programs, the heuristic program possesses the definitely human ability to fail, usually by prematurely eliminating the correct path to the desired conclusion.
>
> The main strategy for evaluating 'promising' goal paths is to work backwards from the goal . . . .

Newell, Shaw, and Simon (1958: 151-166) stated that a program that successfully simulates or models the strategies of the human problem solver may be viewed as a theory that explains human behavior. Two successful programs or theories that seek to explain the human problem-solving behavior by specifying both the needed information and the necessary information-processing operations are The Logic Theorist (LT) and the General Problem Solver (GPS).

The Logic Theorist was a program that was designed by Newell, Shaw, and Simon in 1958. Its purpose was to discover the proofs for the set of theorems in symbolic logic derived by Whitehead and Russell in Principia Mathematica (1925). The computer specialists prepared a set of methods for discovering proofs. These methods were incorporated into LT in the following sequence: substitution, detachment, forward chaining, and finally backward chaining. According to Davis (1973: 65):

> As with the human problem solver, LT's performance critically depended upon the presentation sequence of the cumulatively stored problems; and LT displayed humanlike insight in the sense of employing strategies or heuristics for reducing trial-and-error search behavior.

The General Problem Solver (GPS) was a revision of LT. The new, more powerful program, designed by Newell and Simon (1961: 2011-2017), was capable of playing chess, proving logic theorems and trigonometric identities, solving word puzzles, and even writing computer programs. It could prove or solve any problem that could be recast to include an original state S_o (appropriate formulae and axioms), a goal state S_g (the proof or the solution) and an admissible set of operations. The strategy of GPS was to compare S_o to S_g, obtaining a set of differences. A heuristic program was built into GPS which guided in the search for a sequence of operations that would map S_o to S_g.

In summary, the heuristic programs (a subset of artificial intelligence) closely simulate the human problem-solving processes. In these programs, there is a need to understand the problem, as evidenced in GPS where S_o required the inclusion of the appropriate

formulae and axioms necessary to prove, or to solve the problem. It is necessary in heuristic programs to devise a plan and carry it out as evidenced by the following quotations (Davis, 1973: 65, 66):

> LT displayed humanlike insight in the sense of employing strategies or heuristics for reducing trial-and-error search behavior.
>
> The differences (between S_o and S_g) serve as an heuristic to guide the search for a sequence of operations that would remove those differences. The solution, in fact, is that sequence of operations that 'maps' or moves the original state S_o to the goal state S_g.

Finally, there is a need in heuristic programs to 'look back' over the recently solved problem. That is, the program needs to 'remember' the method that was used to solve a particular problem so that it might be the clue to solving some problem in the future. According to Davis (1973: 61):

> Every computer . . . can even 'learn' in the sense of cumulatively storing new information, instructions, and decisions, and using this newly acquired input to make 'more informed' decisions.

B. Psychological Foundations of the Heuristic Problem-Solving Process

The purpose of this section is to establish a viable psychological foundation for the heuristic problem-solving process. This process was advocated by Polya and adapted for the present study.

The meaning of heuristic was discussed in Chapter I. In summary:

> Heuristic reasoning is reasoning, not regarded as final and strict, but as provisional and plausible only, whose purpose it is to discover the solution of the present problem (Polya, 1957: 113).

The heuristic strategy for solving problems, that Polya advocated in his book, How to Solve It, was documented in the form of a

mathematical problem-solving checklist (Polya, 1957: xvi, xvii). The rationale for this checklist was that a problem solver learns to ask the appropriate question at the appropriate time. That is, there are four main stages through which a problem solver proceeds in order to arrive at the solution to the problem. The four stages are: (1) Understanding the Problem, (2) Devising a Plan, (3) Carrying Out the Plan, and (4) Looking Back. Within each stage, there are a variety of questions to stimulate the individual's mind and, hopefully, lead him to the problem's solution. According to Davis (1973: 117):

> Instead of eliciting a large number of solutions, as in the case of idea checklists, Polya's list teaches different forms of questioning geared to defining and planfully, with imagination, approaching difficult and unfamiliar mathematical tasks.

A question such as: "Do you know a related problem?" is found within the 'Devising a Plan' stage. This was a major triggering mechanism in this experimental study. Once a problem solver had understood the problem he was then faced with devising some plan or strategy for solving it. In order to do this the individual was trained by the teacher to recall a related problem, i.e., a problem with similar characteristics, that he has already solved and to apply this same method of solution to the unsolved problem. Boychuk (1974: 40), in reference to Polya's research, said that students need to acquire a repertoire of problem types that they can recall when confronted with an unsolved problem. She said (1974: 40):

> Polya suggests that the teacher present the student with sample problems from various methods and thereby increases each student's repertoire of responses when he is confronted with a mathematical problem.

Thus, the phase in the heuristic problem-solving process, of

searching the individual's repertoire of problems for a related problem is a major component of the total process. The topics of 'going beyond the given information' and 'transfer of training' provided the psychological basis for this phase, and therefore constituted the conceptual framework for the author's study. Initially these two topics are examined in the following paragraph. Next, discussion of transfer is placed in a developmental psychological context in order to justify the choice of grade eleven students as subjects for this experiment.

Following this, the five methods of solution (generalization, specialization, analogy, decomposing and recombining, and working backward), which the students learned in the process of solving problems, are examined. Each method is defined and then the psychological processes that occur in each of the methods are described.

Finally, these psychological processes are placed in a developmental psychological context to show that Mathematics 20 students, whose approximate age is sixteen years, are developmentally prepared to participate in the experiment.

1. The Heuristic Process

(a) Going Beyond the Information Given

When a problem solver is confronted with a problem, he wants to be able to use his knowledge gained as a result of past experience in solving problems to solve the one presently before him. It is therefore, important that the individual has mentally organized his experiences into appropriate categories (perhaps as in the author's study, on the basis of their method of solution), each with its own

defining properties. According to Bruner (1957: 42):

> (concept development) consists of learning the defining properties of a class of functionally equivalent objects and using the presence of these defining properties as a basis of inferring that a new object encountered is or is not an exemplar of the class.

Once a problem has been placed in a given category, on the basis of its containing some of the basic defining properties of that class, then by inference, the problem is given class identity. Bruner (1957: 42) stated:

> Given the presence of a few defining properties or cues, we go beyond them to the inference of identity.

Having bestowed class identity upon a given problem, a further inference is made; that is, all of the properties of the class can be attributed to the given problem and used in its solution. According to Bruner (1957: 42):

> . . . we infer that the instance so categorized or identified has the other properties characteristic of membership in a category.

Two illustrations may clarify what is meant by 'going beyond the information given.' The first is a general example of concept development; the second example pertains specifically to the present study. An individual picks up an object and notes certain of its sensory properties: its size, shape, and texture. Based upon these cues, he infers that the object is an apple, and hence it is edible, it can be cut with a knife, it rots if left open to the air, and so on.

The second example involves an individual who is attempting to solve a problem. He identifies certain characteristics of the problem; for example, that the outcome of the problem is given and

that it has a trial-and-error beginning. Based upon these properties, he infers that the problem can be placed into the category labelled 'working backward.' Furthermore, the problem can be solved using the method of solution 'working backward.'

Davis (1973: 38) summarized the implications of 'going beyond the information given,' for problem solving, by stating:

> Problem solving should become easier whenever an unfamiliar problem can be identified as a member of a class of problem whose solution-strategy is known . . . .

In the heuristic problem-solving unit contained in the present study, there were five main concepts, the five solution methods, that were developed. The students were taught to place problems in their correct cognitive category. This was done by training the pupils to characterize each category by a few basic properties, and thus to place problems containing these basic properties into the appropriate class. Once a problem was classified, by inference the other attributes of the category (such as its method of solution) were bestowed upon the problem.

(b) Transfer of Training

(i) Definition

Learning is a cumulative process. Every new event that an individual is confronted with is perceived and understood through the 'screen' of past experience. That is, information and habits that one has built up in the past will affect new learning. According to Ellis (1965: 3): "Transfer of training means that experience or performance on one task influences performance on some subsequent task." Individuals who, for example, have learned to operate a

vehicle on the right side of the road, and who have driven that way for several years, find no small difficulty in adjusting to driving conditions in a British-influenced country, where motorists are accustomed to driving on the left side of the road.

A study of transfer is, by definition, a two-stage experiment, comprising a training stage and a test phase. The basic question is: How does the training phase influence performance during the test phase? The training phase may have a positive effect, a negative effect, or no effect on new learning (the test phase). According to Ellis (1965: 3):

> Transfer of learning may take three different forms: (1) performance on one task may aid or facilitate performance on a second task, which represents positive transfer; (2) performance on one task may inhibit or disrupt performance on a second task, which represents negative transfer; and (3) finally, there may be no effect of one task on another, in which case we have an instance of zero transfer.

(ii) Transfer of General Principles

The psychological process, 'transfer of training' (more specifically, transfer of general principles) is a synonymous process with 'going beyond the information.' The reason, then, for mentioning both processes is to provide the reader with a broader understanding of the conceptual framework of this study.

Davis mentioned transfer in connection with Bruner's article "On Going Beyond the Information Given," when he emphasized the need to teach what is generic about a given problem (that is, what is applicable to every member of a class of problems). He stated:

> Incidentally, an important implication for teaching transferrable problem-solving skills is that we should emphasize what is generic about a given problem, in

> order that related problems may be handled more deftly than was the original (Davis, 1973: 39).
>
> Finally, Bruner's conclusions pertaining to "going beyond the information given" directly implied that problem solving may be facilitated by transferring familiar higher-order principles (generic categories theories, and models) to unfamiliar problem situations (Davis, 1973: 39).

In the author's study, the experimental treatment group was taught five general principles (that is, five solution methods), and within each principle, what was generic about the problems. According to Ausubel and Robinson (1969: 154), teaching general principles helps students to transfer information.

> It has long been advocated that lateral transfer can be facilitated by teaching students general principles rather than specific solutions, knowledge, or skills.

When an individual was confronted with a new problem, therefore, the appropriate principle (decided by what is generic about a given problem) was transferred to the problem in order that he could solve it.

Experiments by Judd (1908), Hendrickson and Schroeder (1941), and Brownell and Moser (1949) substantiate the above claim made by Ausubel and Robinson. Hendrickson and Schroeder modified the experiment conducted by Judd, involving learning to hit targets submerged under water. In their study, a target was initially submerged to a depth of six inches. The first group was asked to practice until they could perform the task successfully. The second group was taught the principles of the refraction of light before commencing to practice hitting targets. The third group, in addition to learning the principles of light refraction, was also taught that the angle of refraction increases with the depth of water. After the groups

had mastered the initial task, the target was moved to a new depth (two inches); and the number of attempts needed by each group to master this task, was recorded. The results of their study indicate that the more complete the explanation of the principle, the greater the degree of transfer to the new situation.

(iii) Developmental Psychology and Transfer

Since the author's thesis on heuristic problem solving is founded upon the notion of transfer of training, it would be beneficial to place this phenomenon in a developmental psychological context in order to justify the choice of grade eleven students as subjects for the experiment.

The author will first briefly reinterpret his study on the heuristic approach to problem solving in terms of transfer of training terminology.

The first eleven days of the experiment were occupied with generally teaching the heuristic approach to solving problems and specifically, the five methods of solution needed to utilize the heuristic approach to problem solving. This period was the training phase of a transfer of training experiment. During the last eight days of the study the subjects were presented new problems and asked to solve the problems using the heuristic process. This second period is the testing phase in a transfer of training experiment.

Using the author's strategy of heuristic problem solving, it would be appropriate to examine both the content and the process involved. Generally, the content of the problems used in this study included both algebra and geometry; both of these branches of mathematics transcend particular methods of solution.

More specifically, content usually refers to certain types of problems (for example, age problems and coin problems). One method of solution may be used in solving many types of problems. Similarly, within a given problem-type more than one method of solution may be used. Therefore, subjects who use the problem-type as a cue to determine the method of solution to adopt may be practicing negative transfer. Either they become accustomed to associating a particular problem-type with a certain solution method, in which case the students form a 'mental set' against the associations of the same problem-type with different methods of solution or they 'fixate' on an inappropriate cue (such as content).

Luchins (1942) conducted a series of mental-set experiments with water-jar problems. The subjects' task was to obtain a required volume of water, given specific empty jars for measurement. The experimental subjects were given seven problems (shown in Table 2.1), and were asked to solve them.

PROBLEM	CAPACITY OF EMPTY JARS			REQUIRED CAPACITY
	A	B	C	
#1	21	127	3	100
#2	14	163	25	99
#3	18	43	10	5
#4	9	42	6	21
#5	20	49	4	31
#6	23	49	3	20
#7	15	39	3	18

TABLE 2.1 SEVEN PROBLEMS IN LUCHIN'S EXPERIMENT ON MENTAL SET

Most experimental subjects, after solving the first couple of problems, deduced the formula (B - A - 2C) that consistently, correctly solved problems one to seven. However, most of the experimental subjects did not solve problems six and seven in the simplest possible manner. Whereas the control subjects, who had started with problems six and seven, easily deduced the two respective formulas (A - C) and (A + C) that correctly solved those problems. The explanation, according to Luchins, is that a habit or mental set established in minutes in the psychological laboratory can blind a person to simple solutions.

Duncker (1945), in exploring the process through which people gain insight or fail to gain insight into problems, discovered that 'fixation' on an inappropriate solution prevents many individuals from solving a problem. According to Scheerer (1963: 121): "Duncker discovered that fixation often interferes when the solution of a test problem required the use of a familiar object in a novel way." For example, an individual needs to use a screwdriver in order to solve some pressing problem and one is not available. There is a strong possibility that the person will not be creative enough to consider a viable alternative to the screwdriver, for example, a coin. Although the individual would likely have a coin in his possession, it is unlikely that he would put it into use in this novel manner. The reason for this is that the problem solver thinks of the coin as money and not as a substitute for a screwdriver. Therefore, using Duncker's terminology, the individual fixates on an incorrect course of action and hence fails to solve the problem.

Scheerer (1963) illustrated several causes of fixation in problem solving. One cause is that individuals make incorrect assumptions

about a problem. For example, a person is given nine dots, as shown in Figure 2.1, and asked to connect them, using four straight lines and without lifting the pencil from the paper.

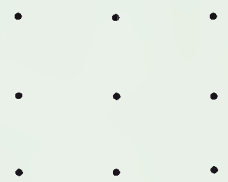


FIGURE 2.1 ILLUSTRATION OF ONE CAUSE OF FIXATION

The problem is solved by extending the lines beyond the dots which the problem solver either did not think of doing or assumed that he could not do. Another illustration is the problem in which the individual is asked to construct four equilateral triangles, given six matches. The solution is to construct a pyramid with a triangle-shaped base. Again the individual likely assumed that the solution must lie on a plane.

Another cause of fixation in problem solving is that individuals may fail to detect an object's suitability for a problem because the needed object is hidden in a conventional context. An example is the ring-and-peg problem shown in Figure 2.2

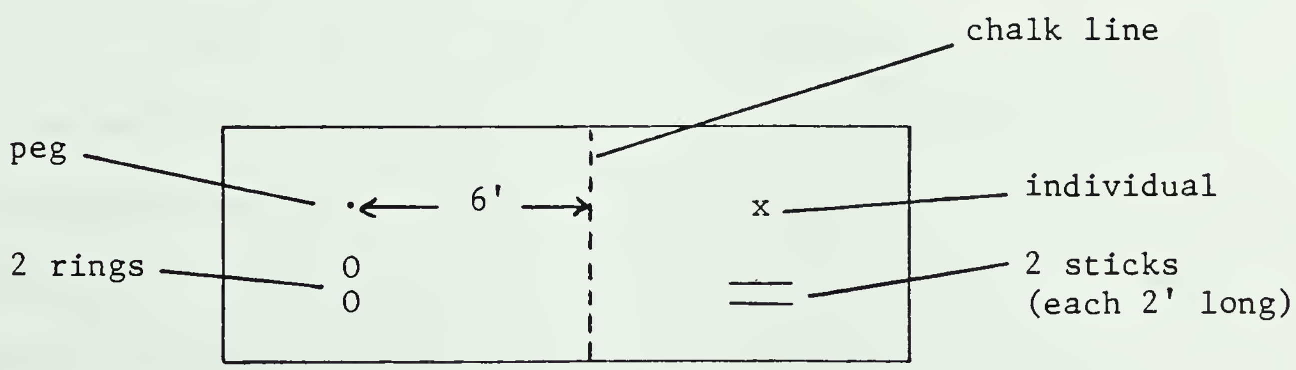


FIGURE 2.2 ILLUSTRATION OF ONE CAUSE OF FIXATION

To solve the problem, the individual should first use a string to

tie the two sticks together thus getting one stick approximately four feet long, then using the newly constructed stick he should reach over the chalkline, spear the two rings, and place them on the peg. During the experiment itself, three groups were presented with a different situation. The string was left alone on a nail in one situation, it was used to hang up an old, worn calendar in another, and it was used as a hanger for functional things like a clear mirror in the third situation. The results were that most subjects thought of using the string in the first situation, whereas less than half of them thought of using the string in the third situation.

The heuristic process begins with an attempt, on the part of the problem solver, to understand the problem. During this initial stage, the individual notes explicitly, what are the given data and what is the unknown. Once the problem is clearly understood, he begins to devise a plan to solve the problem. To help him choose the correct method of solution to use in a given problem, he might examine the given data or the unknown which in turn might help him to recall another similar problem that he has already solved, whose method of solution is appropriate in the present problem.

If the content of the problem or the components of the problem, i.e., the data and the unknown, are not helpful cues in discovering an appropriate method of solution, then the individual should systematically consider each of the five methods of solution. This approach to devising a plan can be used by a problem solver in approaching every problem; however, it is more cumbersome and takes longer.

The approach to devising a plan, which entails a systematic examination of all the possible methods of solution, exemplifies the essential characteristic of Piaget's stage of formal-operational thought. According to Flavell (1963: 204):

> The most important general property of formal-operational thought, the one from which Piaget derives all others, concerns the real versus the possible. Unlike the concrete-operational child, the adolescent begins his consideration of the problem at hand by trying to envisage all the possible relations which could hold true in the data and then attempts, through a combination of experimentation and logical analysis, to find out which of these possible relations in fact do hold true.

The need for the subjects to utilize formal-operational thinking is evident because all five methods of solution are possible, but not all five are realistic for any given problem. Hopefully, the subjects through a process of logical analysis will discard all unrealistic solution methods and focus their attention only upon the realistic possibilities. Thus, the essence of the heuristic process is hypothetico-deductive thinking; that is, reducing the amount of mechanical trial-and-error searching by selectively pursuing only goal paths that appear promising.

2. Developmental Psychology and the Five Methods of Solution

The major intention of the study reported here was to translate Polya's heuristic approach to problem solving into a unit on heuristic problem solving to be given to grade eleven (Mathematics 20) students. In the process of developing a teaching strategy for training students in this heuristic process the author was forced to make some delimitations. One delimitation was to curtail the number of methods of solution that the instructor taught the students. Polya's book, How

to Solve It (1957), gives eight methods of solution (generalization, specialization, analogy, decomposing and recombining, working backwards, definition, recursion, and auxiliary elements and problems) that can be used in problem solving. Of these eight methods of solution, the author chose five (generalization, specialization, analogy, decomposing and recombining, and working backwards). Definition was not included because this method was rarely used by itself in solving a problem, but usually was utilized only as a prerequisite to complete solution by clarifying the problem so that a plan for solving the problem could be devised. Recursion was not included because the application of this method of solution to mathematics problems was beyond the grasp of the average high school student. Finally, auxiliary elements and problems was not included because this method of solution also does not exist independent of the other methods of solution. For example, this method is used in setting up equations which are included within another method of solution, decomposing and recombining.

This subsection will be devoted to a discussion of these five methods of solution beginning with a brief definition of each method. Then the psychological processes that occur in the methods will be described. Finally, these psychological processes, typifying the corresponding methods of solution, will be placed in a developmental psychological context. Before doing this, a general comment should be made with regard to the level of thinking that is necessary in order to solve word problems.

By their very nature, word problems are an abstraction of the

real world. In other words, the objects that word problems make reference to are not themselves present for the student to manipulate. Therefore, the problem solver must 'reflect' these operations in the absence of the objects. This is the essence of formal-operational thought, which is necessary in order to solve word problems given the absence of manipulable objects. According to Piaget (1967:63):

> What, in effect, are the conditions for the construction of formal thought? The child must not only apply operations to objects—in other words, mentally execute possible actions on them—he must also 'reflect' these operations in the absence of the objects which are replaced by pure propositions.

Generalization is a method of solution used by Polya in his heuristic approach to problem solving. Basically, it involves a consideration of specific instances (from which a pattern is observed) to an inference that what holds true for the specific cases will also hold true for 'n'; thus a generalization. Ausubel (1958: 552) said that as children grow older, they tend to perceive, think, and organize their cognitive worlds in increasingly more general terms. According to Ausubel:

> Another expression of this tendency is shown in the growing trend to attribute properties to objects and situations on the basis of inference (generalization) rather than on the basis of direct experience.

Generalization can occur in the concrete-operational stage as inference made from the perception or manipulation of concrete objects. However, since word problems are one level removed from the concrete situation, the problem solver is operating at the formal-operational stage by drawing implications from propositions. According to Piaget (1967: 63):

> Formal operations engender a 'logic of propositions' in contrast to the logic of relations, classes, and numbers engendered by concrete operations. The system of 'implications' that governs these propositions is merely an abstract translation of the system of 'inference' that governs concrete operations.

Specialization is another method of solution used by Polya in his heuristic approach to problem solving. Briefly, it entails the consideration of some specialized case which either solves the problem or provides the strategy to solve the problem. According to Ausubel (1958: 570):

> Young children profit less from hints and are less able to generalize or transpose solutions to more abstract and remote situations.

Choosing the appropriate specialized case requires that the problem solver think of a simpler but related situation to the given problem. Even if the word problem was represented with concrete materials, finding a specialized case would still require the individual to reflect on a simpler but related situation. According to Piaget (1967: 63):

> Formal operations provide thinking with an entirely new ability that detaches and liberates thinking from concrete reality and permits it to build its own reflections.

Also, choosing the appropriate specialized case requires constant mental reflection back to the given problem. Piaget (1967: 64):

> The free activity of spontaneous reflection is one of the two essential innovations that distinguish adolescence from childhood.

Analogy is another method of solution used by Polya in his heuristic approach to problem solving. Basically, it involves a consideration of two <u>systems</u> which agree in clearly definable relations between their respective parts. For instance, a triangle in a plane

is analogous to a tetrahedron in space. These two analogous systems, whether they are in an algebraic form or a geometric form, are not set before the individual as concrete objects but exist only as reflections in his mind. Not only are the analogous systems reflections in the problem-solver's mind, but the relationships that exist between the two systems are also reflections. According to Piaget (1967: 63):

> This reflection is thought raised to the second power.
>
> Concrete thinking is the representation of a possible action, and formal thinking is the representation of a representation of possible action.

Therefore, word problems needing the analogy method for solution require that the individual be operating at the formal-operational level of thinking.

Decomposing and recombining is another method of solution used by Polya in his heuristic approach to problem solving. It entails decomposing a problem into its component parts and then recombining the parts into a more or less different whole. The problems in the author's study that utilize this method for their solution require the individual to break the word problem into parts (phrases). These phrases are then translated into mathematical symbols (further abstraction), which are recombined into mathematical sentences (equations). According to Piaget (1967: 62, 63):

> As of eleven to twelve years, formal thinking becomes possible, i.e., the logical operations begin to be transposed from the plane of concrete manipulation to the ideational plane, where they are expressed in some kind of language (words, mathematical symbols, etc.), without the support of perception, experience, or even faith.

These mathematical symbols and mathematical sentences are abstract statements which contain the data from the problem. These statements represent examples of propositional thinking. According to Flavell (1958: 205):

> Formal thinking is above all propositional thinking. The important entities which the adolescent manipulates in his reasoning are no longer the raw reality data themselves, but assertions or statements—propositions—which 'contain' these data.

Thus, decomposing and recombining is a method of solution employed by individuals who are operating at the formal-operational level of thinking.

Working backwards is another method of solution used by Polya in his heuristic approach to problem solving. It involves starting from what is required, and then assuming what is sought as already found. From this assumption, the individual works backward from the 'solved' problem to the initial state in which the problem was presented. Having thus discovered the solution to the problem in a retrogressive manner, it remains to reverse the process and start from the point that the individual reached last in the analysis.

Working backwards entails removing oneself to a higher level of abstraction when the individual assumes the problem as solved. It also requires the consideration of the possible (assuming the problem is solved) versus the real (the problem in its original state). According to Flavell (1958: 205): ". . . the adolescent has, through his new orientation, the potentiality of imagining all that might be there . . . ." These two properties of working backward, reflection and the real versus the possible, are also two important

characteristics of formal-operational thought. Thus, a problem solver, who is required to use the working backwards method in solving a word problem, needs to be operating at the formal-operational level of thinking.

CHAPTER III

DESIGN OF THE STUDY

The purpose of this chapter is to provide preliminary information for the construction and validation of measuring instruments (Chapter IV) and the heuristic problem-solving strategy (Chapter V). This chapter includes the research procedures and design that were established, the research questions and hypotheses that guided the study, and the statistical techniques that were utilized in answering the research questions.

A. Research Procedures and Design

1. Sample

The subjects utilized in this research study were grade XI (Mathematics 20) students who belong to the Edmonton Public School System. Altogether, two classes were involved in the experiment: both classes were situated in the same school.

2. Pilot Study of Experimental Treatment

A preliminary study was conducted in October 1975 with a grade IX class in the Edmonton Public School System. The purpose of the six-day study was to provide the author with an opportunity of ascertaining whether or not it was possible to implement Polya's heuristic approach to problem solving. The study indicated that, indeed, it was possible.

The pilot study was administered to a Mathematics 20 class of thirty-one pupils in the Edmonton Public School System. This study was conducted over a three-week period, from November 23, 1976 until

December 13, 1976. The purpose of this pilot project was three-fold: one, to ensure that the strategical development of each of the lesson plans could be followed by the students; two, to determine whether the fifteen lessons of the experimental treatment could be taught within a forty-minute time limit; three, to determine whether the problems contained in the fifteen lessons were suitable both in terms of their level of difficulty and their uniqueness of solution method. As a result of this pilot study, numerous corrections were made to the lesson plans of the experimental treatment.

The cooperating teacher, after observing all of the lessons, consented to teach the experimental treatment to his Mathematics 20 class in May 1977, as a part of the research study.

Typical student comments, taken from the Student Opinion Survey, revealed their enthusiasm for this problem-solving unit. For example, in response to question four in the survey ("Do you feel that you are a better problem solver now than before? Why?"), they commented:

> Yes, because before I didn't know how to solve problems in such an organized manner. In fact, I didn't know how to do them, period.
>
> Yes I think so because now when I look at a problem I can easily characterize it and I will know what method to use to solve it.

3. Research Procedures

The author was given access to two Mathematics 20 classes, both located in the same school, in order to perform an experimental study of nineteen days duration, more precisely, sixteen days for forty minutes per day and three days for eighty minutes per day.

Due to the practical difficulty of finding a cooperating teacher

who: (1) had two Mathematics 20 classes; (2) would have been willing to teach both treatments, the author conducted both the experimental and control treatments himself.

4. Research Design

Diagrammatically, the design of this research study was as follows:

EXPERIMENTAL GROUP	0	X_1	0 0 0
CONTROL GROUP	0	X_2	0 0

The 0 stands for making observations (that is, measurements) and the X represents the application of the treatment. A broken horizontal line between the rows representing the different groups indicates that the two groups were not equivalent in either a matching or a random sampling sense at the beginning of the investigation.

This research study was conducted over nineteen days. Tests were administered on the first, tenth, seventeenth, eighteenth, and nineteenth days. The treatments were administered during the interim.

5. Research Study

(a) Experimental Treatment (Outline of Lesson Plans)

To be more explicit, a topical outline of the nineteen lesson plans, in the experimental study, is listed below:

DAY	TIME ALLOTMENT	TOPIC
1	80 min.	Pretest - Mathematics 20 Problem-Solving Test
2	40 min.	Teach Method of Generalization
3	40 min.	Teach Method of Generalization
4	40 min.	Teach Method of Working Backward
5	40 min.	Teach Method of Working Backward

6	40 min.	Teach Method of Specialization
7	40 min.	Teach Method of Analogy
8	40 min.	Teach Method of Decomposing and Recombining
9	40 min.	Teach Method of Decomposing and Recombining
10	40 min.	Five-Method Comprehension Test
11	40 min.	Discussion of Test
12	40 min.	Mixture of Problems
13	40 min.	Mixture of Problems
14	40 min.	Mixture of Problems
15	40 min.	Characteristics and Solution-Methods Quiz
16	40 min.	Discussion of Quiz and Open Discussion
17	80 min.	Heuristic Problem-Solving Test
18	20 min.	Characteristics and Solution-Methods Test
	20 min.	Student Opinion Survey
19	80 min.	Post-test - Mathematics 20 Problem-Solving Test

TABLE 3.1 OUTLINE OF LESSON PLANS FOR THE EXPERIMENTAL TREATMENT

During lessons two through nine, the experimental treatment group was taught the five methods of solution (Generalization, Working Backward, Specialization, Analogy, and Decomposing and Recombining). They were taught the general steps of the heuristic problem-solving process, how to use each method of solution, as well as the characteristics of problems employing particular methods of solution. The purpose of lessons ten and eleven was to answer the question: Have the students learned to correctly use the five methods of solution to solve problems? The instructor presented them with five problems and told them which method of solution to use for each problem. It was the students' responsibility to use each given method to solve the five problems. One day was allotted for the Five-Method Comprehension Test and the

next day was utilized for a discussion of their mistakes. Throughout lessons twelve through fourteen, the students learned to choose one of the five methods of solution in order to solve some unknown problem that they were given. The purpose of lessons fifteen and sixteen was to answer the question: Have the students learned to choose the appropriate solution method for a given problem? The teacher endeavoured, during these two lessons, to crystallize in the students' minds, the associations that were established between problem characteristics and their matching method of solution. The pupils were presented with ten problems, two problems for each of the five solution methods, and asked to write down the characteristics of each problem and its appropriate solution method. One day was allotted for the Characteristics and Solution-Methods Quiz and the next day was utilized for a discussion of their mistakes.

It should be noted that different methods of solution were allotted different lengths of instruction time. That is, Specialization and Analogy were each allotted one day; Generalization, Working Backward and Decomposing and Recombining were each assigned two days. The author chose the respective time allotments on the basis of the prevalence of these types of problems in mathematics textbooks and research documents.

(b) Control Treatment

A conventional problem-solving process was taught to the control group. As the word 'conventional' denotes, this process represented a typical approach that is used in teaching students to solve problems. The same problems were used with the control groups as were used with

the experimental groups, but their order of presentation was different; that is, all twenty-four problems were categorized into five groups (according to their methods of solution) and each group was taught in 'isolation.'

PROBLEM GROUP	LESSON
#1	Lessons 2-6
#2	Lessons 6-8
#3	Lessons 8-10
#4	Lessons 10-12
#5	Lessons 12-16

TABLE 3.2 FIVE GROUPS OF PROBLEMS FOR THE CONTROL TREATMENT

The schedule that was followed in teaching the conventional problem-solving process is outlined in the table below:

DAY	TIME ALLOTMENT	TOPIC
1	80 min.	Pretest - Mathematics 20 Problem-Solving Test
2	40 min.	Give out the booklet of 24 problems. Do #1. Assign #2 for homework.
3	40 min.	Correct #2. Assign #3, #4.
4	40 min.	Correct #3, #4. Assign #5, #6.
5	40 min.	Correct #5, #6. Assign #7.
6	40 min.	Correct #7. Do #8. Assign #9, #10.
7	40 min.	Correct #9, #10. Assign #11.

8	40 min.	Correct #11. Do #12. Assign #13.
9	40 min.	Correct #13. Assign #14.
10	40 min.	Correct #14. Do #15. Assign #16.
11	40 min.	Correct #16. Assign #17.
12	40 min.	Correct #17. Do #18. Assign #19.
13	40 min.	Correct #19. Assign #20, #21.
14	40 min.	Correct #20, #21. Assign #22, #23.
15	40 min.	Correct #22, #23. Assign #24.
16	40 min.	Correct #24. Open discussion.
17	80 min.	Heuristic Problem-Solving Test
18	20 min.	Student Opinion Survey
19	80 min.	Post-test - Mathematics 20 Problem-Solving Test

TABLE 3.3 OUTLINE OF LESSON PLANS FOR THE CONTROL TREATMENT

The instructional strategy employed in this process was to assign the problem for homework on one day and present them with their solutions on the next school day. The instructor discussed each solution with the class noting the logic and the mechanics that were utilized in solving that particular problem but did not mention either the name of the method of solution employed or other solution methods that were used in solving other problems and how those solution methods were related to the present problem.

The solutions that the students were given for the assigned homework problems were the same as the solutions given to the corresponding problems in the experimental treatment, with the following exceptions: first, the general steps of the heuristic problem-solving

process, i.e., Understanding the Problem, Devising a Plan, Carrying Out the Plan, and Looking Back, with their accompanying questions, as advocated by Polya, were not included; secondly, no mention was made of the characteristics of any problem and no descriptive names, as to problem-type, were provided other than the labels groups #1, #2, #3, #4, and #5.

6. Implementation of Treatments

	EXPERIMENTAL TREATMENT	CONTROL TREATMENT
Mathematics Course Utilized	Mathematics 20 course	
Number of Classes	One	One
Treatment Administrator	Researcher	Researcher
Dates Conducted	May 2, 1977- May 27, 1977	May 2, 1977- May 27, 1977
Daily Time Block	1. 8:30 A.M.-9:50 A.M. 2. 9:10 A.M.-9:50 A.M.	1:50 P.M.-3:10 P.M. 2:30 P.M.-3:10 P.M.

TABLE 3.4 DESCRIPTIVE INFORMATION OF THE EXPERIMENTAL GROUP AND THE CONTROL GROUP

(a) Experimental Treatment

The experimental treatment for this research study was administered by the author to one class of Mathematics 20 students located in the Edmonton Public School System. The month of May was considered an appropriate time because, due to the semester system, the students would be at the same place in the Mathematics 20 curriculum as the pilot class was in the first semester.

The author came into the Mathematics 20 class beginning on May 2, 1977 and ending on May 27, 1977. Altogether, the experimental treatment occupied nineteen days (sixteen forty-minute periods and three eighty-minute periods). The forty-minute lessons were taught during the second half of the eighty-minute block.

(b) Control Treatment

The control treatment for this research study was administered by the author to one class of Mathematics 20 students in the Edmonton Public School System. Nineteen lessons were conducted (seventeen forty-minute periods and two eighty-minute periods) from May 2, 1977 until May 27, 1977. The small time difference, forty minutes, between the two treatment groups occurred as a result of one less test being administered to the control group during the eighteenth lesson.

This Mathematics 20 class was also a semestered class. In addition, both the experimental group and the control group had covered similar material and were at the same place in the Mathematics 20 curriculum at the start of the main study. During the study, both groups were taught similar concepts from the Mathematics 20 curriculum by their respective cooperating teachers. This was done during the first forty minutes of their double periods, thus preventing either of the two classes from getting behind in their normal program.

B. Research Questions and Hypotheses

The purpose of this section is to translate the general questions, given in the Statement of the Problem, Chapter I, Section B of this thesis, into scientific hypotheses which could be tested. Following each question, there are one or more scientific hypotheses. Subsequent

to each of these predictions is its justification, which has utilized the results of research studies, psychological theory, and logic as its basis. Finally a null hypothesis or a set of criteria has followed each scientific hypothesis, by which the author could ultimately answer each of the three questions. Some of the scientific hypotheses could not be tested utilizing an inferential statistical technique, therefore a set of criteria was established as a basis for nominally testing them.

1. Question One

Statement: "What attitudes do students and teachers have toward a heuristic problem-solving instructional strategy?"

Scientific Hypothesis #1: "The students and teacher will have a positive attitude toward a heuristic problem-solving instructional strategy."

Data on the attitudes of both cooperating teacher and students were gathered by means of the Cooperating Teacher Opinion Survey and the Student Opinion Survey respectively.

Set of Criteria for

Scientific Hypothesis #1: The researcher had established a level of 70% or more responding "yes" to each of the following questions on the Student Opinion Survey:

1. "Did you find that the procedure presented was helpful in solving problems?"

2. "Do you feel that you are a better problem solver now than before?"
3. "Do you think that this unit will help you in solving new problems in Mathematics 20?"

The researcher had established that a strong declaration by the cooperating teacher to the question: "Do you feel that the students benefited by learning to use this method of problem solving?" was an indication of positive attitude toward the instructional strategy.

2. Question Two

Statement: "Does such an instructional strategy affect the students' ability in solving those problems which are peculiar to those utilized in that strategy?"

Scientific Hypothesis #2: "It would seem probable that students who have been subjected to the heuristic problem-solving treatment, will successfully use given methods of solution in solving problems."

The level of learning that was required of the experimental group in this activity was of an elementary nature. The students were taught to use a particular method of solution to solve certain problems. They were then given a problem, and told to use that particular

method of solution in solving it. That is, the teacher demonstrated a method of solution on one problem, the students imitated this procedure on several other problems, and subsequently the students were tested on the Five-Method Comprehension Test to ensure that they could correctly use the particular methods of solution.

Perhaps the most difficult stage in solving any problem is in identifying the problem as a member of a class of problems whose solution method is known. In the Five-Method Comprehension Test, the answers were provided for that stage. The only requirement was to correctly utilize the given solution methods in solving the problems on the test.

Set of Criteria for

Scientific Hypothesis #2:

1. The Five-Method Comprehension Test (page 86). The researcher had established an individual mean of 70% or above as an indication of successful use of a given method of solution in solving the problem.
2. The Heuristic Problem-Solving Test (pages 86-88). The researcher had established an individual mean of 40% or above as an indication of successful use of a given method of solution in solving the problem.

Scientific Hypothesis #3: "It would seem probable that students, who have been subjected to the

heuristic problem-solving treatment, will perform better on a heuristic problem-solving test, than students who have been taught using a conventional problem-solving treatment."

The test problems themselves offered no advantages to either treatment group, since both groups were trained on the same problems. The experimental group, however, was taught the general steps of the heuristic problem-solving process by which every problem was solved, as well as how to characterize problems and to associate them with their appropriate solution methods; whereas the control group was given no such training. Therefore, the experimental group had the definite advantage of having been explicitly taught to develop concepts.

Null Hypothesis for
Scientific Hypothesis #3: "There is no significant difference in mean scores between the instructional methods as measured by the Heuristic Problem-Solving Test."

Scientific Hypothesis #4: "It would seem probable that there is a strong relationship between scores on a heuristic problem-solving test and scores on a characteristics and solution-methods test, for students who have been subjected to the heuristic problem-solving treatment."

The heuristic problem-solving treatment in the researcher's study is

based upon the process of concept development. Jarolimek (1966: 534) has defined a concept in the following manner: "Concepts may be regarded as categories of meaning." He (1966: 534) further stated that: "Concept development calls for the placing of information in correct cognitive categories." In the heuristic problem-solving treatment, there are five main concepts that were developed; these being the five solution methods. The students were taught to place problems in their correct cognitive category. This was done by training the pupils to characterize each category by a few basic properties, and thus to place problems containing these basic properties into the appropriate class.

Once a problem had been classified, by inference the other attributes of the category, e.g., its method of solution, were assigned to the problem. Bruner, Goodnow, and Austin (1962: 244) described concepts and their development as:

> . . . a network of sign-significate inferences by which one goes beyond a set of observed criterial properties exhibited by an object or event to the class identity of the object or event in question, and then to additional inferences about the unobserved properties of the object or event.

The Heuristic Problem-Solving Test was designed to measure the students' ability to choose the correct solution method and thus to solve the problem. Although it is thought that the pupils would do this by: first, characterizing the problem; second, choosing the appropriate method of solution; and third, solving the problem, it was not essential that the students either characterize the problems or use any of the five methods of solution in solving them. However,

the literature on concept development, as noted in this section and in Chapter II of this thesis, indicates that placement of problems into their correct cognitive categories occurs as a result of the matching of characteristics of problems with the criterial properties of particular categories. Thus, there is substantial evidence in the literature to legitimately hypothesize that the students would do the Heuristic Problem-Solving Test by characterizing each problem, then choosing the appropriate solution method, and finally, solving the problem. Therefore, there should be a high correlation between the scores on the Heuristic Problem-Solving Test and the scores on the Characteristics and Solution-Methods Test for students who had been subjected to the heuristic problem-solving treatment.

Null Hypothesis for
Scientific Hypothesis #4: The correlation coefficient between the scores on the Heuristic Problem-Solving Test and the Characteristics and Solution-Methods Test is zero.

3. Question Three

Statement: "Does such an instructional strategy affect the students' ability in solving those problems which are peculiar to those utilized at a specific grade and course level."

Scientific Hypothesis #5: "It would seem probable that students, who have been subjected to the heuristic problem-solving treatment will perform better on a Mathematics 20

problem-solving test, than students who have been taught using a conventional problem-solving treatment."

Scientific Hypothesis #6: "It would seem probable that students, who have been subjected to the heuristic problem-solving treatment, will perform better on the post-test of a Mathematics 20 problem-solving test, than they did on its pretest."

Although Lucas' (1974) study was performed on first-year university students, his results indicated that students who were trained in heuristic processes obtained more accurate results than those who were not. Ashton's (1962) study revealed similar results, but was performed on grade IX mathematics students. It was hoped that further support for this hypothesis would result from the heuristic problem-solving treatment located in Appendix G of this thesis. In this method of instruction two processes are taught to the students. The first is the general steps of the heuristic problem-solving process by which every problem was solved. These general steps are the basic framework of this method of instruction, into which the second process is placed. This second process involves five methods of solution, which were employed to solve the problems in this research study. Either the first process or the second process, or their combined systematic approach to problem solving, will markedly improve the students' problem-solving test scores.

Null Hypothesis for

Scientific Hypothesis #5: "There is no significant difference in mean scores between the two instructional methods as measured by the Mathematics 20 Problem-Solving Test."

Null Hypothesis for

Scientific Hypothesis #6: "There is no significant mean difference between the scores on the pretest and the post-test of the Mathematics 20 Problem-Solving Test, for the heuristic problem-solving treatment group."

C. Statistical Procedures

The statistical procedures utilized in this research study in analyzing the results were as follows:

1. On the null hypothesis for scientific hypothesis #3 (page 56), a two-way analysis of variance was performed on the scores obtained from the Heuristic Problem-Solving Test. The two factors that were utilized in this analysis were group levels and treatments.
2. For scientific hypothesis #4 (page 58), the Pearson product-moment correlation coefficient between the scores on the Heuristic Problem-Solving Test and the Characteristics and Solution-Methods Test was calculated. The Fisher r to Z transformation was then utilized in determining the significance of the obtained Pearson product-moment correlation coefficient.

3. On the null hypothesis for scientific hypothesis #5 (page 60), a two-way analysis of variance was performed on the scores obtained from the Mathematics 20 Problem-Solving Test. The two factors that were utilized in the analysis were group levels and treatments.
4. On the null hypothesis for scientific hypothesis #6 (page 60), a one-way analysis of variance was performed on the scores obtained from the pretest and post-test of the Mathematics 20 Problem-Solving Test, for the heuristic problem-solving treatment group.

CHAPTER IV

THE HEURISTIC PROBLEM-SOLVING METHOD OF INSTRUCTION

In this method of instruction two simultaneous processes were taught to the students. The first was the general steps of the heuristic problem-solving process by which every problem was solved. These general steps were the basic framework of this method of instruction, into which the second process was placed. This second process involved five methods of solution, which were employed to solve the problems in this research study.

A. General Steps of the Heuristic Process

The procedure, that was followed in teaching the students how to solve problems using this process, began with the teacher solving the first problem. As he worked through the problem he verbalized his thoughts, in order for the students to hear the process that he used and the one that they would be asked to follow in solving problems themselves. After the teacher had demonstrated how to solve a problem using a particular method of solution, he asked the students to try the next problem. They imitated the method that the teacher used to solve the previous problem. In presenting new methods of solution, the basic format of this method of instruction is demonstration followed by imitation.

In solving a problem the problem solver was taught to always follow the same systematic approach. This approach to solving problems involved working through four basic stages: Understanding

the Problem, Devising a Plan, Carrying Out the Plan, and Looking Back. Within each stage, the students were taught to ask questions appropriate to that stage. These general steps of the heuristic process with their accompanying questions were advocated by Polya in his book, How to Solve It (1957), and are documented there in the form of a mathematical problem-solving checklist.

1. Understanding the Problem

Understanding the problem is the first step in the heuristic problem-solving process. This step is the one in which two questions are raised and answers to them are sought. These questions are:

> What is the unknown?
> What information is given?

2. Devising a Plan

After the problem was understood, i.e., the unknown and the data had been delineated, it remained to devise some plan by which the problem solver might proceed from the given data to the unknown. The question that students were taught to ask themselves, in this second step of the heuristic process was: Do you know a related problem? The purpose of this question was to get the problem solver to discover a method of solution that had a reasonable chance of leading to the correct solution. Before responding to this question, the teacher named the characteristics of the problem currently before him. He then thought of a related problem with similar characteristics and named its method of solution. The reason for matching problems in this way was to allow the problem solver to use the same method of solution to solve the problem whose solution was unknown. This is heuristic reasoning, whose purpose it is to discover the solution of

the present problem through the means of a reasonable guess.

Once the method of solution was decided, another question was asked to introduce the designated method of solution. This question depended upon the method of solution that was chosen as having the greatest plausibility of achieving the correct answer. For example, if generalization was the designated method, then the question that introduces it was: Do you know a related problem? If specialization was the chosen method, then the appropriate question that introduces this method of solution was: Do you know a more specialized problem? If analogy was the designated method, then the question that introduces it was: Do you know an analogous problem? If decomposing and recombining was the chosen method, then the appropriate question that introduces this method of solution was: Can you restate the problem? The problem was restated in the form of a mathematical sentence. If working backward was the designated method, then the question that introduces it was: Can you restate the problem? The problem was restated as if it was already solved.

3. Carrying Out the Plan

This third stage involved going back to the devised plan and carrying out the necessary computations in order to solve for the unknown. The student was taught to check to ensure that he had carried out the plan carefully and accurately.

4. Looking Back

At the fourth and final stage the problem solver was taught to examine the solution that had been attained. The students were taught to ask questions and seek answers that were appropriate for this stage.

These questions were:

> Is the result correct?
>
> What are the characteristics of this problem that could be associated with the method of solution that was utilized?

In response to the second question, the students were required to write down the characteristics of the problem and the method of solution used in correctly solving the problem. The reason for explicitly noting this association between problem characteristics and solution method was to help the students, in the future, when another problem would be encountered and they must decide which of the five methods of solution to use in order to solve that problem.

B. Five Methods of Solution

The other process that was taught to the subjects concurrently with the first process, the General Steps of the Heuristic Process, was the Five Methods of Solution to be utilized in solving the problems. A good understanding of each of the five methods of solution, and the characteristics of problems that were associated with each particular method of solution was essential before the problem solver could properly fulfill the second stage of the heuristic process.

In the lesson plans for this heuristic problem-solving process, lessons two through eleven were devoted to the teaching of these methods of solution. Lessons twelve through sixteen allowed the students to put into practice all they had learned in the previous lessons. The students were presented with problems whose method of solution was unknown, and were asked to solve the problems working through each step of the heuristic process.

1. Generalization

Generalization is a method of solution used by Polya in his heuristic approach to problem solving. It involves a consideration of specific instances from which a pattern is observed to an inference that what holds true for the specific cases will also hold true for 'n'; thus a generalization.

For example, given the problem:

> In a triangle ABC, if 20 lines are drawn from the vertex A through points on the opposite size, how many triangles are formed?

The instructional strategy that was used in solving this problem follows. The teacher wrote the problem on the blackboard and then paused for several minutes allowing the students sufficient time to understand the problem. He then began by asking: What is the unknown? He answered that the unknown is the resultant number of triangles that are formed. The second question was: What information is given? The given data were: a triangle ABC and 20 lines drawn from vertex A through points on the opposite side. Since the unknown and the data had been delineated, it remained to devise some plan by which the instructor might proceed from the given data to the unknown.

The teacher then asked the following question: Do you know a related problem? The purpose of this question was to get the problem solver to discover a method of solution that had a reasonable chance of leading to the correct solution. Before responding to this question, the teacher named the characteristics of the problem currently before him. He thought of a related problem with similar characteristics and named its method of solution. Since this problem

was characterized as a problem involving pattern development, the method of solution decided upon by the instructor was generalization.

The teacher then asked himself the question: Do you know a related problem? A simple related problem that he gave was:

> If 0 lines are drawn from vertex A through 0 points of the opposite side, how many triangles are formed?

He then asked if there was another related problem that would help in solving the original problem. The answer that the instructor gave was to present the second related problem:

> If 1 line is drawn from vertex A through 1 point of the opposite side, how many triangles are formed?

He then asked if there was another related problem that would help in solving the original problem. The answer the teacher gave was to present the third related problem:

> If 2 lines are drawn from vertex A through 2 points of the opposite side, how many triangles are formed?

He then asked if there was another related problem that would help in solving the original problem. The answer the instructor gave was to present the fourth related problem:

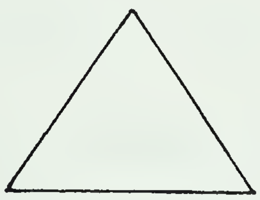
> If 3 lines are drawn from vertex A through 3 points of the opposite side, how many triangles are formed?

Next the teacher asked if these four related problems could help them in solving the original problem. He drew the table as shown below and asked if the students could see a pattern developing that would lead to the solution of the problem.

Number of Lines	Number of Triangles
0	1
1	3
2	6
3	10
.	.
.	.
.	.
20	?

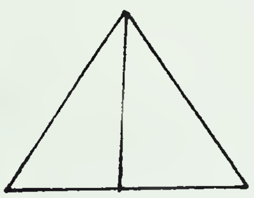
It should be noted that pictures of triangles corresponding to each of the four related problems were drawn on the blackboard as each related problem was presented. This aided in clarifying the four related problems used in devising the plan.

Case #1 -



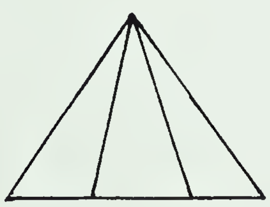
If 0 lines are drawn from vertex A through 0 points of the opposite side, there is 1 triangle that is formed.

Case #2 -

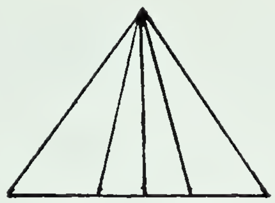


If 1 line is drawn from vertex A through 1 point of the opposite side, there are 3 triangles that are formed.

Case #3 -



If 2 lines are drawn from vertex A through 2 points of the opposite side, there are 6 triangles that are formed.

Case #4 -  If 3 lines are drawn from vertex A through 3 points of the opposite side, there are 10 triangles that are formed.

The teacher then expanded the table to include all the possibilities 0-20, for the number of lines drawn.

After the plan was carried out and a solution was reached, the teacher asked the question:

> Is the result correct?
>
> What are the characteristics of this problem that could be associated with the method of solution that was utilized?

The instructor first checked his result to ensure that it was correct. After he did that, he examined the problem again in order to note its characteristics. He then wrote them on the blackboard along with the method of solution used to correctly solve this problem. The characteristics were:

1. The number of lines is large; the estimated answer is very large.

2. To draw a diagram depicting the situation, and to attempt to count all the resulting triangles is an extremely difficult task.

2. Specialization

Specialization is another method of solution used by Polya in his heuristic approach to problem solving. Briefly, it entails the consideration of some specializaed case which either, in itself, solves the problem, or provides the strategy to solve the problem.

For example, given the problem:

> Bob has 10 pockets and 44 silver dollars. He wants to put his dollars into his pockets, so distributed

> that each pocket contains a different number of dollars. Can he do so?

The instructional strategy that was used in solving this problem follows. The teacher wrote the problem on the blackboard and then paused for several minutes allowing the students sufficient time to understand the problem. The instructor then began by asking: What is the unknown? He answered that the unknown was the distribution of the 44 coins into 10 pockets so that each pocket contained a different number of coins. The second question that he asked was: What information is given? He answered that the given data were 44 coins, 10 pockets, and a distribution constraint for the coins (a different number of coins are to be placed in each pocket).

Since the unknown and the data were delineated, it remained to devise some plan by which the teacher might proceed from the given data to the unknown. The instructor then asked the following question: Do you know a related problem? The purpose of this question was to get the problem solver to discover a method of solution that had a reasonable chance of leading to the correct solution. Before responding to that question, the teacher named the characteristics of the problem currently before him. He then thought of a related problem with similar characteristics and named its method of solution. Since this problem was characterized as a problem that focused on a particular case, that of finding the minimum number of silver dollars necessary to satisfy the requirement of the problem, the method of solution decided upon by the teacher was specialization.

The instructor then asked the question: Do you know a more specialized problem? The teacher then wrote the special problem on

the blackboard.

> What is the minimum number of silver dollars that could be placed into 10 pockets?

He then solved the more specialized problem and thus demonstrated that it was impossible for Bob to place a different number of coins into each of 10 pockets, given that he only had 44 coins.

First pocket gets 0 coins

Second pocket gets 1 coin

Third pocket gets 2 coins

.

.

.

Tenth pocket gets 9 coins

The minimum number of coins = 0 + 1 + 2 + . . . + 9 = 45 coins. Therefore, since Bob only had 44 coins, he could not put a different number into each pocket.

After the plan was carried out and a solution was reached, the teacher asked the questions:

> Is the result correct?
>
> What are the characteristics of this problem that could be associated with the method of solution that was utilized?

The instructor first checked his result to ensure that it was correct. After he did that, he examined the problem again in order to note its characteristics. He wrote them on the blackboard along with the method of solution used to correctly solve this problem. The characteristics were:

1. A decision needs to be made between two alternatives (Can he do it or can he not do it?).

2. A specialized problem could give a clue as to the choice (to be made between the two alternatives) or decide the choice.

3. Analogy

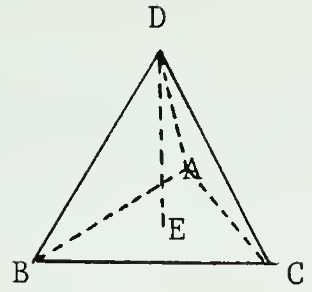
Analogy is another method of solution used by Polya in his heuristic approach to problem solving. Basically, it involves a consideration of two systems which agree in clearly definable relations between their respective parts. For instance, a triangle in a plane is analogous to a tetrahedron in space.

For example, given the problem:

> Find the altitude of a regular tetrahedron with given edge 10 cm. (Hint: the foot of the altitude divides a median of the triangular base into two line segments whose lengths are in the ratio 2:1.)

The instructional strategy that was used in solving this problem follows. The teacher wrote the problem on the blackboard and then paused for several minutes allowing the students sufficient time to understand the problem. The instructor then began by asking: What is the unknown? He answered that the unknown was the altitude of the regular tetrahedron. The second question was: What information is given? The given data were: A tetrahedron with edge length 10 cm, and that the foot of the altitude divides a median of the triangular base into two line segments whose lengths are in the ratio 2:1.

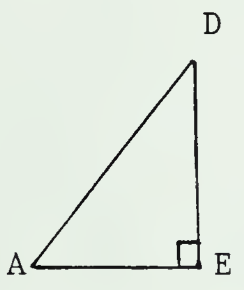
Since the unknown and the data were delineated, it remained to devise some plan by which the teacher might proceed from the given to the unknown. The instructor began by drawing a diagram depicting the original situation.



He then asked the question: Do you know a related problem? The purpose of this question was to have the students discover a method of solution which had a reasonable chance of leading to the correct solution. Before responding to this question, the teacher named the characteristics of the problem currently before him. He then thought of a related problem with similar characteristics and named its solution method. Since this problem was characterized as a problem involving a three-dimensional figure, which asked for a one-dimensional answer (i.e., the altitude of the regular tetrahedron), the method of solution decided upon by the teacher was analogy.

The instructor then asked the question: Do you know an analogous problem? The answer was to write an analogous problem on the blackboard. He then proceeded to solve the analogous problem.

> Given a right triangle ADE and given the length of the sides $\overline{AD}$ and $\overline{AE}$; find the length of the side $\overline{DE}$.



Pythagoras' theorem states:

$(\overline{AD})^2 = (\overline{AE})^2 + (\overline{DE})^2$. Therefore,

$(\overline{DE}) = \sqrt{(\overline{AD})^2 - (\overline{AE})^2}$.

The procedure that he used in solving this analogous problem was helpful in guiding him through the solution to the original problem.

First:

Triangle ADE is a right triangle (because $\overline{DE}$ is the altitude).

Therefore $\overline{AD}^2 = \overline{AE}^2 + \overline{DE}^2$ (Pythagoras' theorem)

or $\overline{DE} = \sqrt{\overline{AD}^2 - \overline{AE}^2}$

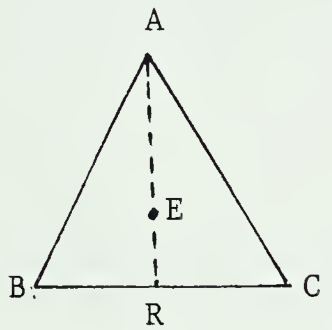
but $\overline{AD} = 10$ cm (edge of tetrahedron)

Therefore $\overline{DE} = \sqrt{100 - \overline{AE}^2}$

Second:

Find $\overline{AE}$.

Examine triangle ABC.



Given altitude $\overline{AR}$

$\overline{ER} = \frac{1}{2}\overline{AE}$

Find $\overline{AR}$.

Examine right angle ARC.

Pythagoras theorem states:

$$\overline{AC}^2 = \overline{RC}^2 + \overline{AR}^2$$

$$10^2 = 5^2 + \overline{AR}^2$$

$$AR = \sqrt{10^2 - 5^2}$$

$$= 5\sqrt{3} \text{ cm}$$

but $\overline{AR} = \overline{AE} + \overline{ER} = 5\sqrt{3}$ cm

$= \overline{AE} + \frac{1}{2}\overline{AE} = 5\sqrt{3}$ cm

Therefore $(3/2)(\overline{AE}) = 5\sqrt{3}$ cm

$\overline{AE} = 10\sqrt{3}/3$ cm

Third:

$$DE = \sqrt{100 - \overline{AE}^2}$$

$$= \sqrt{100 - 100(3)/9}$$

$$= \sqrt{(900-300)/9}$$

$$= \sqrt{600/9}$$

$$= 10\sqrt{6/3} \text{ cm}$$

$$= 8.16 \text{ cm}$$

Therefore, the height of the regular tetrahedron was 8.16 cm.

After the plan was carried out and a solution was reached, the instructor asked the questions:

> Is the result correct?
>
> What are characteristics of this problem that could be associated with the method of solution that was utilized?

The teacher first checked his result to ensure that it was correct. After he did that, he examined the problem again in order to note its characteristics. He wrote them on the blackboard along with the method of solution used to correctly solve this problem. The characteristics were:

1. The problem involves a three-dimensional figure (a regular tetrahedron) but asks for a one-dimensional answer (altitude).
2. Using a triangle (a two-dimensional figure) to find the altitude of a regular tetrahedron (a three-dimensional figure) appears to be a necessity. The triangle in a plane is analogous to a tetrahedron in space.

4. Decomposing and Recombining

Decomposing and recombining is another method of solution used by Polya in his heuristic approach to problem solving. Briefly, it entails decomposing the problem into its component parts and then recombining the parts into a more or less different whole. For

example, given the problem:

> A man spends 18% of his monthly salary for rent. If his rent is $45 a month, what is his salary for a month?

The instructional strategy that was used in solving this problem follows. The teacher wrote the problem on the blackboard and then paused for several minutes allowing the students sufficient time to understand the problem. The instructor then began by asking: What is the unknown? He answered that the unknown was the salary for one month, and he wrote this on the blackboard. The second question was: What information is given? The given data are that 18% of the salary is for rent and that the rent is $45 per month.

Since the unknown and the data were delineated, it remained to devise some plan by which the teacher might proceed from the given data to the unknown. He then asked himself the question: Do you know a related problem? The purpose of this question was to have the students discover a method of solution which had a reasonable chance of leading to the correct solution. Before responding to this question, the teacher named the characteristics of the problem currently before him. He then thought of a related problem with similar characteristics and named its method of solution. Since this problem was characterized as a problem in which a mathematics sentence could be established, the method of solution decided upon by the teacher was decomposing and recombining.

The instructor then asked himself the question: Can you restate the problem? The answer that the teacher gave was to restate the problem in such a way that he progressed closer to the solution to this problem. In this case, the instructor set up a mathematics

sentence, expressing in mathematical symbols, a condition that was stated in words. The equivalent mathematics sentence was: $(.18)(x) = 45$, where x is the salary for one month. He then solved the equation for x and thus found the value of the unknown number.

Let x be the salary per month in dollars.

Original Problem	Algebraic Translation
A man spends 18% of his monthly Salary for rent.	.18
If his rent is $45 a month,	45
what is his salary for a month?	x

$(.18)(x) = 45$ or $(18)/(100) = (45)/(x)$

$x = 45/(.18)$

$= 250$

Therefore, his salary for a month is $250.

After the plan was carried out and a solution was reached, the teacher asked the questions:

> Is the result correct?
>
> What are the characteristics of this problem that could be associated with the method of solution that was utilized?

The instructor first checked his result to ensure that it was correct. After he did that, he examined the problem again in order to note its characteristics. He wrote them on the blackboard along with the method of solution used to correctly solve this problem. The characteristics were:

1. A problem in which one comparison is made between his rent and salary; thus one equation can be established in order to generate the solution.

2. A problem which is easily translated into one mathematics setnence.

5. <u>Working Backward</u>

Working backward is another method of solution used by Polya in his heuristic approach to problem solving. Basically, it involves starting from what is required and then assuming what is sought as already found. From this assumption, the individual works backward from the 'solved' problem to the initial state in which the problem was presented. Having thus discovered the solution to the problem in a retrogressive manner, it remains to reverse the process and start from the point which the individual reached last in the analysis.

For example, given the problem:

> Given a jar that will hold exactly 9 quarts of water, a jar that will hold exactly 4 quarts of water, no other containers holding water, but an infinite supply of water, describe a sequence of fillings and emptyings of water jars that will result in achieving 6 quarts of water.

The instructional strategy that was used in solving this problem follows. The teacher wrote the problem on the blackboard and then paused for several minutes allowing the students sufficient time to understand the problem. The instructor began by asking: What is the unknown? He answered that the unknown was to fill the nine-quart pail with six quarts of water. The second question was: What information is given? The given data are: an unlimited supply of water, a four-quart pail (unmarked with any quantity scale), and a nine-quart pail (also unmarked).

Since the unknown and the data were delineated, it remained to devise some plan by which the instructor might proceed from the given data to the unknown. The teacher then asked the following question: Do you know a related problem? The purpose of this question was to get the problem solver to discover a method of solution that had a reasonable chance of leading to the correct solution. Before responding to that question, the teacher named the characteristics of the problem currently before him. He then thought of a related problem with similar characteristics and named its method of solution. Since this problem was characterized as a problem in which there were a large number of paths that led away from the initial state but only one that reached the given goal state, the method of solution decided upon by the teacher was working backward.

The instructor then asked the question: Can you restate the problem? The answer that the teacher gave was to restate the problem:

> Given that you have a four-quart pail and a nine-quart pail, and also given that your nine-quart pail already has six quarts of water in it, how did you get it there?

The instructor then started at the end of the solution and worked backward. Questions similar to the following, with corresponding replies, were used to devise a plan or strategy.

1. Question: "If you have six quarts of water in the nine-quart pail, where did it come from?"

   Reply: "The water came from the nine quarts of water in the full pail of which three quarts were removed."

2. Question: "How were the three quarts removed?"

   Reply: "The water (three quarts) was poured out of

the full nine-quart pail into the four-quart pail in which there was already one quart."

3. Question: "Where did the one quart of water, in the four-quart pail, come from?"

Reply: "The one quart of water came as a result of pouring out eight quarts of water (filling the four-quart pail twice) from a full nine-quart pail. Then pour the remaining one quart into the empty four-quart pail."

Now that a plan was devised, the teacher carried it out, starting at the beginning of the problem. Take a nine-quart pail, full of water, and pour out eight quarts (filling the four-quart pail twice). Then pour the remaining one quart into the empty four-quart pail. Next, refill the nine-quart pail. Pour out the water from the nine-quart pail to fill the four-quart pail. There are now only six quarts remaining in the nine-quart pail. It should be noted that pictures of pails, with their corresponding quantities of water, were drawn on the blackboard at appropriate times throughout the problem. This aided in clarifying the questions and their replies used in devising a plan.

After the plan was carried out and a solution was reached, the instructor asked the questions:

> Is the result correct?
>
> What are the characteristics of this problem that could be associated with the method of solution that was utilized?

The teacher first checked his result to ensure that it was correct. After he did that, he examined the problem again in order to note its characteristics. He wrote them on the blackboard along with the method of solution used to correctly solve this problem. The

characteristics were:

1. The outcome is known (six quarts of water).

2. The problem has a trial-and-error beginning.

The five examples given in this section were problems extracted from the researcher's lesson plans for the experimental treatment. It should be noted that these five problems are typical first problems in their respective lesson plans and thus they are the problems whose solutions were demonstrated to the class. Subsequent problems within each of their respective lesson plans involved students attempting the solutions themselves by imitating the teacher's demonstration of the first problems.

C. The Heuristic Problem-Solving Method of Instruction

The purpose of this section is to summarize the heuristic problem-solving process. The two processes which were taught simultaneously were the:

1. General Steps of the Heuristic Problem-Solving Process
   - (a) Understanding the Problem
   - (b) Devising a Plan
   - (c) Carrying Out the Plan
   - (d) Looking Back
2. Five Methods of Solution
   - (a) Generalization
   - (b) Working Backward
   - (c) Specialization
   - (d) Analogy

(e) Decomposing and Recombining.

The students were taught to ask appropriate questions throughout the heuristic problem-solving process, which were intended to guide them through the solution to each problem. The students were taught the appropriate questions to ask themselves but were never taught the names of the four steps of the heuristic process. Table 5.1 contains the general steps of the heuristic process along with the appropriate questions and the students' responses at each step.

GENERAL STEPS OF THE HEURISTIC PROCESS	APPROPRIATE QUESTIONS THAT STUDENTS ASK THEMSELVES	EXAMPLES OF STUDENTS' RESPONSES
Understanding the Problem	What are the unknowns?	Write the unknowns.
	What is given?	Write the given information.
Devising a Plan	Do you know a related problem?	Examine problem and <u>name</u> its characteristics.
		Think of a related problem (with similar characteristics) whose solution method is known.
		Name the method of solution.
Carrying Out the Plan	Do you know a related problem?	This question initiates the method of generalization.
(Now have <u>one</u> solution method to use, with reasonable certainty of getting the correct answer.)	Can you restate the problem (as solved)?	Restating the problem as solved initiates the method of working backward.
	Do you know a more specialized problem?	This question initiates the method of specialization.
	Do you know an analogous problem?	This question initiates the method of analogy.
	Can you restate the problem (as mathematical sentences)?	Restating the problem as mathematical sentences initiates the method of decomposing and recombining.
Looking Back	Is the result correct?	Students will check their solution.
	What are characteristics and solution method?	The students will <u>write</u> down the characteristics of the problem and the name of its method of solution.

TABLE 4.1 THE GENERAL STEPS OF THE HEURISTIC PROCESS ALONG WITH APPROPRIATE QUESTIONS AND STUDENTS' RESPONSES TO EACH STEP

CHAPTER V

CONSTRUCTION AND VALIDATION OF MEASURING INSTRUMENTS

A. Introduction

Four tests were utilized in this research study as measuring instruments. The first instrument was the Mathematics 20 Problem-Solving Test; it was used as a pretest (in lesson one) and as a post-test (in lesson nineteen). This test was designed to measure the problem-solving level of grade XI (Mathematics 20) students.

The second instrument was the Five-Method Comprehension Test; it was administered in lesson ten after the pupils had completed the previous eight lessons in which they were taught the five methods of solution. This test was designed to measure the students' competence in successfully using given methods of solution.

The third instrument was the Heuristic Problem-Solving Test; it was administered in lesson seventeen. The purpose of this test was to measure how well the pupils solved problems similar to those given in the two treatments (experimental and control). Whereas the problems in this test arose directly from the treatments, the problems in the Mathematics 20 Problem-Solving Test were only indirectly associated with the problems in the treatments.

Although the problems contained in the third instrument were similar to those used in the treatments, it was not essential that the students follow the methods of solution that they had been taught in the treatments, in order to score full marks on that test.

Thus, there was a need to know the extent to which the pupils could explicitly and directly use the information taught them in the treatments. That was the purpose of the fourth instrument, the Characteristics and Solution Methods Test. It was administered in lesson eighteen.

The fifth and sixth instruments that were utilized in this research study were the Student Opinion Survey and the Teacher Opinion Survey. The Student Opinion Survey was administered to the students in lesson eighteen, following the Characteristics and Solution-Methods Test and the Teacher Opinion Survey was given to the teachers following the completion of lesson sixteen.

The six instruments that were used in this study are included in Appendices A-F of this thesis.

B. Instrumentation

1. Development of the Instruments

(a) Mathematics 20 Problem-Solving Test

This instrument was initially administered at a high school in the Edmonton Separate School System on October 28, 1976. There were thirty-three Mathematics 20 students who wrote the test. The grand mean was 36.2%, and the mean scores on each of the seven problems were 53%, 48%, 73%, 49%, 1%, 13%, and 16% respectively.

On the basis of these results and the author's classroom observations, the following decisions were made. First, problem six would be eliminated from the test (because of its low mean, 13%, and because the test was too long). Secondly, the author decided to modify problem five. Although its mean was 1%, it was not eliminated because

it represented the geometry content in the Mathematics 20 curriculum. The modifications took the form of four hints, which either provided definitions, formulas, or necessary mathematical background information, but did not explicitly provide clues as to the problem-solving strategy. The third decision was to have a timed test. The reason for this was to ensure that the students would attempt all six problems. Fourthly, one hint was given for problem seven (because its mean score was 16%).

The modified form of this instrument was administered to a Mathematics 20 class in the Edmonton Public School System, on November 2, 1976. Based on this testing experience (with thirty-three students), two more decisions were made with regard to the feasibility of the test. First, this instrument could not be administered until the students had completed the section (in the Mathematics 20 curriculum) on systems of equations, in particular, two equations and two unknowns. Secondly, the hint provided with problem seven as a result of the initial test administration was unnecessary. Evidently, it was not the difficulty level of that problem which caused its 16% mean, but rather a shortage of time.

On November 18, 1976 the third administration of this instrument took place. It was conducted with twenty Mathematics 20 students in the Edmonton Public School System. The grand mean was 39.8%, and the six individual problem means were 49%, 21%, 51%, 40%, 28%, and 50%. The author's suspicions from the second testing were confirmed by this testing: the test was still too long and that problem two was too difficult for the Mathematics 20 students. It had also become

evident from the three test administrations, that the students were able to get the correct answers to problems three and four simply by trial and error. These two problems were modified in order to reduce this random trial-and-error guessing. The modified problems contained the same subject content but the numbers were larger and the resulting mathematics sentences were more complex.

The Mathematics 20 Problem-Solving Test, in its modified form, was administered during the week of December 6, 1976 to two classes (forty-five pupils) of Mathematics 20 students in the Edmonton Separate School System. The grand mean was 43.6%, and the five individual problem means were 40%, 63%, 65%, 22%, and 29%. This instrument is found in Appendix A of this thesis.

(b) Five-Method Comprehension Test

This test was administered to a mathematics class of thirty-three students, on December 3, 1976, in the Edmonton Public School System. The administration of this test was during the pilot study phase of the experimental treatment. The grand mean was 73.5% and the five individual problem means (each problem required a different method of solution) were 84%, 70%, 64%, 62%, and 88%.

The five problems contained in this test were taken directly from the section of the treatment that the students had already completed. Since this was a comprehension test, the five problems were altered so that students would not be able to simply memorize the problem solutions from lessons two to nine, and reproduce them on this test. This instrument is located in Appendix B of this thesis.

(c) Heuristic Problem-Solving Test

This instrument was initially administered, on November 2, 1976,

to thirty-three Mathematics 20 students in the Edmonton Separate School System. Although this instrument was designed to follow the treatment, it was felt worthwhile to administer the test without having given the pupils prior training.

This test had seven problems, two of which involved decomposing and recombining as their method of solution, two utilized generalization, and the other three problems required three different methods of solution: specialization, analogy or working backward. As a result of the initial test administration, a number of decisions were made. First, the test was too long. As in the Mathematics 20 Problem-Solving Test, five problems were sufficient for an eighty-minute period. Therefore, problem one and problem three were eliminated (thus leaving five problems, each requiring a different method of solution). Secondly, problem two (specialization-type solution) and problem five (decomposing and recombining-type solution) were to be left intact. Third, problem six (generalization-type solution) required further clarification and therefore two definitional hints were added. The fourth decision that was made as a result of the initial administration of this instrument was to replace problem four (working backward-type solution) and problem seven (analogy-type solution) with respectively similar solution-type problems. The reason for this replacement was that two other problems that better represented these two methods of solution were obtained.

On December 14, 1976, following the administration of the experimental treatment at a high school in the Edmonton Public School System, the modified version of the Heuristic Problem-Solving Test was

administered to a Mathematics 20 class of twenty-nine students. The grand mean was 54.1% and the five individual problem means were 51% (decomposing and recombining-type problem), 27% (specialization-type problem), 81% (working backward-type problem), 52% (analogy-type problem), and 59% (generalization-type problem). The 27% mean for problem two indicated that further clarification should be given to the appropriate section in the experimental treatment but that the problem itself should remain intact. This instrument is located in Appendix C of this thesis.

(d) <u>Characteristics and Solution-Methods Test</u>

A pilot study of the experimental treatment was conducted, over a three-week period in November and December of 1976, in the Edmonton Public School System. Lesson fifteen, in this treatment, consists of a quiz involving ten previously-solved problems, in which the students were asked to write down both the characteristics of each problem and their corresponding method of solution. The directions given at the beginning of the quiz and a twenty-minute time limit were both adequate in enabling the students to successfully do the quiz (grand mean was 86.9%). Therefore, similar directions and time limit were adopted for the Characteristics and Solution-Methods Test. This instrument is located in Appendix D of this thesis.

(e) <u>Student Opinion Survey</u>

This instrument was administered to thirty-one students following the completion of the three-week pilot study of the experimental treatment. The administration and results of this survey indicated that twenty minutes was sufficient time to complete the six questions,

and that the pupils clearly understood the meaning of each of the questions. This instrument is located in Appendix E of this thesis.

(f) Teacher Opinion Survey

This instrument is a modified form of the "Cooperating Teacher Follow-Up Questionnaire" that the author had previously developed for his colloquium paper. This instrument is located in Appendix F of this thesis.

2. Validity

Webster's New Collegiate Dictionary (1951: 940) suggests that a test (or anything else) is valid if it is "Founded on truth or fact; capable of being justified, supported or defended; well grounded; sound." Lindquist (1942: 213), in defining validity specifically in its relationship to tests, said:

> The validity of a test may be defined as the accuracy with which it measures that which it is intended to measure, or as the degree to which it approaches infallibility in measuring what it purports to measure.

In this research study, there were four instruments which needed to be examined in terms of their validity. The four tests were the Mathematics 20 Problem-Solving Test, the Five-Method Comprehension Test, the Heuristic Problem-Solving Test, and the Characteristics and Solution-Methods Test. It was necessary to examine the appropriateness and content validity of each of the four instruments as well as the construct validity of the fourth instrument. What is meant by appropriateness of the instruments is the extent to which the students have already studied the prerequisite mathematics skills needed for each test. The construct validity for each of the other

three instruments was not examined for the following reasons: (1) the students were permitted to solve the problems contained in the Heuristic Problem-Solving Test and the Mathematics 20 Problem-Solving Test utilizing any means they wished; (2) the Five-Method Comprehension Test required that the students simply apply given solution methods to each of the problems.

(a) Appropriateness of the Instruments

In developing the Mathematics 20 Problem-Solving Test, the author chose problems whose requisite mathematics skills had already been studied by the students. In addition, he also chose an appropriate proportion of geometry and algebra problems.

There are eight units in the Edmonton Public School Board Mathematics 20 curriculum and they are given in the following order: Radicals, Relations, Logarithms, Quadratic Equations, Equations, and Inequations, Systems of Equations, Complex Numbers, Trigonometry, and Geometry. Therefore, the Mathematics 20 Problem-Solving Test should include about 20% geometry and about 80% algebra, since one of the eight units in the Mathematics 20 curriculum is on geometry. The research study was conducted in the classrooms at the completion of unit V (Systems of Equations) and therefore the unit on geometry (unit VIII) had not yet been discussed. Thus, the geometry content was taken from both a unit in Mathematics 9 and a unit in Mathematics 10. The appropriate objectives were:

26. Solve volume problems for right circular cones given diagrams and word problems. (Mathematics 9, unit III, objective #26)

12. Students should be able to do numerical problems based on similar triangles. (Mathematics 10, unit V, objective #12)

Traditionally, unit IV (Quadratic Functions, Equations, and Inequations) and unit V (Systems of Equations) comprise the mathematics skills in any Mathematics 20 (Algebra) problem-solving examination. However, the author discovered that in two of the schools in which he had piloted this test, the instructors had taught systems of equations before teaching quadratics. Therefore, the author decided to use unit V (Systems of Equations) as the algebraic contribution to the Mathematics 20 Problem-Solving Test, but to exclude those objectives which involved either graphing (Mathematics 20, unit V, objectives #1(a), #2, #3, and #6), quadratics (Mathematics 20, unit V, objective #7), or three equations and three unknowns (Mathematics 20, unit V, objective #5). The remaining objectives were:

1. The student should be able to maintain previously developed skills and ideas:
   (b) Solving linear and quadratic equations

4. The student should be able to solve, algebraically, linear systems in two variables.

Problems one, two, and three in the Mathematics 20 Problem-Solving Test utilized unit V, objective #4 of the Mathematics 20 curriculum. Problem four (geometry problem) utilized unit III, objective #26 of the Mathematics 9 curriculum and unit V, objective #12 of the Mathematics 10 curriculum. Problem five utilized unit V, objective #1 of the Mathematics 20 curriculum.

The other three instruments were directly related to the treatment, which focussed entirely upon the teaching of process. Therefore, with one exception, these instruments required only arithmetic computational skills studied in the primary grades. The one exception was the Heuristic Problem-Solving Test which contained one problem

requiring a knowledge of Pythagoras' Theorem. This concept was studied in grade IX as evidenced by the objectives:

> Mathematics IX, Unit 3 - Geometry
>
> 5. State the Pythagorean Property.
> 6. State the formula $c^2 = a^2 + b^2$.
> 8. Use the Pythagorean Property to solve problems given diagrams and word problems.

(b) Content Validity

Before examining the tests (individually) in order to verify their content validity, two complementary definitions of this validity-type should be noted:

> Content validity is concerned with the adequacy of sampling of a specified universe of content. (American Educational Research Association, 1955: 16)

> Finding the content validity of a measuring instrument is equivalent to showing how well it samples certain types of situations or subject matter. The instrument claiming high content validity clearly attempts to include a cross-sectional sample of a great variety of items representing the area in which the pupils performance is being measured. (Ahmann and Glock, 1967: 287)

Traditionally, the universe of content for any Mathematics 20 (Algebra) problem-solving examination includes the following problem-types: age-type problems, coin-type problems, digit-type problems, distance-type problems, and mixture-type problems. In developing the Mathematics 20 Problem-Solving Test, there were six problems; five algebra problems, one per problem-type and one geometry problem. However, because eighty minutes was insufficient time to complete the test, one problem was excluded. Since the mixture-type problem required a knowledge of solving quadratic equations, which some students had not yet studied, it was the problem that was excluded from the test.

Problems one to five of the Mathematics 20 Problem-Solving Test sampled the following respective problem-types: age problem, coin problem, digit problem, geometry problem, and distance problem. Therefore, since the five problems of this test adequately sample the universe of content from which they were developed, there is justification in stating that the Mathematics 20 Problem-Solving Test has a high content validity.

The second instrument, the Five-Method Comprehension Test, was designed as a representative sample of the following specific objective:

> Given a problem and given the name of its solution method, the pupil should be able to correctly solve the problem.

The test was composed of five problems, each requiring a different method of solution. Since there were only five solution methods taught to the students and since the name of each method was written beside the appropriate problem (on the test), then there is justification in stating that the Five-Method Comprehension Test has a high content validity.

The third instrument, the Heuristic Problem-Solving Test, was designed as a sampling of the following specific objective:

> Given a problem (whose solution method has already been taught to the students), the pupil should be able to choose the correct solution method and thus solve the problem.

This test was composed of five problems, each representing a solution method. The students were taught to correctly characterize a given problem and thus to choose the appropriate method of solution. They were also taught to correctly use each solution method in solving the appropriate problems. However, if a pupil chose to correctly solve the problems using some solution method other than those taught to

him, then he still scored full marks. Since the five problems of the Heuristic Problem-Solving Test are an adequate sample of the specific objective, then this instrument has a high content validity.

The fourth instrument, the Characteristics and Solution-Methods Test, was designed to measure the two processes underlying the specific objective of the third instrument. Since it was permissible for the pupils to solve the five problems contained in the third instrument, using their own methods, it was still necessary to measure their knowledge of these two underlying processes. The two specific objectives were:

1. Given a problem (whose solution method has been taught to the students), the pupil should be able to correctly characterize it.

2. Given a problem (whose solution method has been taught to the students), the pupil should be able to correctly choose the associated method of solution.

The same five problems were given in this test, as were given in the Heuristic Problem-Solving Test; in addition, there were five new problems. For each problem, the pupil was asked to characterize it and to choose the correct solution method. Since the ten problems (that is, two problems for each solution method) contained in the Characteristics and Solution-Methods Test were an adequate sample of the two specific objectives, then that is justification for stating that this test has a high content validity.

(c) Construct Validity of the Characteristics and Solution-Methods Test

Ahmann and Glock (1967: 289) defined a construct as ". . . a characteristic assumed to exist to account for some aspect of human behavior." Thus, to refer to the construct validity of the Characteristics and Solution-Methods Test is to assess the accuracy with which

this instrument measures the construct it purports to measure. Cronbach and Meehl (1955), whose study on construct validity is probably the most widely known, elaborated and clarified the concept. Ahmann and Glock (1967: 302) summarized their article by stating:

> The basis on which the investigation of construct validity proceeds is provided by the theory underlying the construct supposedly involved in the measuring instrument. On the basis of the theory, predictions are made . . . . The evaluation instrument is then used to test the predictions. If the predictions and the data produced by the instrument concur, evidence in support of the construct validity has been found. If they do not, a state of uncertainty exists. Either the instrument does not involve the construct, or the theory is not sound. In any event, the degree of construct validity is in doubt. (The underlining is the researcher's.)

The author utilized the three stated requirements of construct validity, as summarized by Ahmann and Glock, as a framework upon which he attempted to justify the construct validity of the Characteristics and Solution-Methods Test.

(i) Theory Underlying the Construct

Ahmann and Glock (1967: 289) refer to the "ability to apply principles" as an illustration of a construct in psychology. The Characteristics and Solution-Methods Test was designed to measure the construct 'ability to choose correct principles.' In particular, here and in what follows, the construct measured by this test was the ability to choose among five solution methods that were taught in the experimental treatment. The theory of concept development which underlies this construct is found in this thesis in Chapter III, pages 56-58 and in Chapter II, pages 27-31. The theory reveals that the construct 'ability to choose correct principles,' is an essential component of the problem-solving process.

(ii) Prediction of Characteristics and Solution-Methods Test Based upon the Theory Underlying the Construct

On the basis of the theory, Ahmann and Glock (1967: 302) stated that a prediction as to whether or not the instrument measured the construct, could be made. Before making this prediction, it is necessary to discuss the Characteristics and Solution-Methods Test and its relationship to the construct.

The theory of concept development underlies the construct 'ability to choose correct principles,' proposedly measured by the Characteristics and Solution-Methods Test. Bruner, Goodnow, and Austin (1962: 244) summarized the literature on concept development in the following excerpt from their book:

> . . . a network of sign-significate inferences by which one goes beyond a set of observed criterial properties exhibited by an object or event to the class identity of the object or event in question, and then to additional inferences about the unobserved properties of the object or event.

They defined the three stages in concept development as: identifying the criterial properties, identifying the class to which the object or event belongs, and making additional inferences about the unobserved properties of the object or event.

The Characteristics and Solution-Methods Test contains ten problems, two problems for each of the five methods of solution. The two specific objectives of this instrument were:

1. Given a problem (whose solution method has been taught to the students), the pupil should be able to correctly characterize it.

2. Given a problem (whose solution method has been taught to the students), the pupil should be able to correctly choose the associated method of solution.

The first requirement of this instrument was that the students characterize each of the ten problems. That is the first stage of concept development; that of identifying the criterial properties.

The second requirement of this instrument was that the pupils identify the appropriate solution method of each of the ten problems. That is the second stage of concept development; that of class identity.

Because the class is identified by the name of a method of solution, that same name also reveals an appropriate unobserved property of the class; that is, an actual solution strategy to follow in order to solve the problem. Thus, the third stage of concept development is also contained in stating the name of a method of solution.

On the basis of the theory, Ahmann and Glock (1967: 302) stated that a prediction must be made. Because the theory revealed that the construct 'ability to choose correct principles' is related to the Characteristics and Solution-Methods Test and is an essential component of the problem-solving process, the author would predict that there would be a significant correlation between the Characteristics and Solution-Methods Test and the Heuristic Problem-Solving Test.

(iii) Evaluation of the Prediction

Finally, Ahmann and Glock (1967: 302) stated that the evaluation instrument is then utilized to test the prediction. According to Helmstadter (1970: 310), this evaluation can take the form of providing experimental or empirical evidence.

> . . . and finally, securing data which will empirically or experimentally confirm or reject the hypothesis.

The Heuristic Problem-Solving Test does measure problem-solving performance. Since the theory revealed that the construct 'ability

to choose correct principles' is an essential component of the problem-solving process, therefore the Heuristic Problem-Solving Test does measure this construct.

Table 6.16 contains the Pearson product-moment correlation coefficient for the Heuristic Problem-Solving Test with the Characteristics and Solution-Methods Test. It is 0.58. Utilizing Fisher's r to Z transformation formula (Hays, 1963:531) to test whether r = 0.58 is significantly different from 0, a correlation coefficient of 0.45 was calculated as the minimum necessary for significance. Therefore there is a significant relationship between these two tests. The coefficient indicates that the construct 'ability to choose correct principles' was a significant component of both tests. Since the prediction and the datum support one another, there is evidence in support of the construct validity of this test.

3. Reliability

In order to clarify the reader's understanding of the concept of reliability, Ebel (1965: 310) contrasted its definition with that of validity. He said:

> The term "reliability" means the consistency with which a set of test scores measure whatever they do measure.
>
> The term "validity" means the accuracy with which a set of test scores measure what they ought to measure.

This research study utilized six instruments for collecting data, but only one of these is a pretest: the Mathematics 20 Problem-Solving Test. Therefore, the primary purpose of this section is to give the reliability coefficient of this test. The reliability

coefficients of the remaining five instruments were not found for the following reasons: (1) two of the five instruments were opinion surveys; (2) the other three instruments contained insufficient questions, they required that at least part of the treatment be taught before the tests could be attempted, and they measured more than one factor.

To obtain this reliability coefficient, it was necessary to collect two independent sets of scores on this test from the members of the same group. The method that the researcher used for obtaining these two independent measurements was the "stable method" (Ahmann and Glock, 1967: 315) or, as it is more commonly known, the "test-retest method." The "parallel forms method" was not chosen because the problem-solving unit that had been developed in this research study had been designed for classroom use; therefore a single problem-solving test was more practical. The "split-halves method" for obtaining two sets of independent scores was rejected because of the limited number of problems that could be solved in an eighty-minute period. (The Mathematics 20 Problem-Solving Test contains five problems and has a seventy-minute time allotment.) The "Kuder-Richardson method" of measuring the internal consistency (or homogeneity) of a test was unacceptable because it is premised on the assumption that the test only measures a single factor. Aiken (1973), in a literature review of problem-solving ability, listed five predominant factors that arose from factor-analytic investigations. They are: deductive reasoning, inductive reasoning, numerical ability, spatial-perceptual ability, and verbal comprehension.

The two test administrations were conducted in two classes of Mathematics 20 students in the Edmonton Separate School System. The first testing was conducted during the week of December 6, 1976 and the retest was administered on January 4, 1977. In order to guard against obtaining a faulty reliability coefficient, the following precautions were taken:

1. The cooperating teacher was asked not to discuss the contents of the test with his students or to inform them about the retest, in the interim between the test and the retest.

2. A four-week interim was set to prevent the students from recalling many of their previous responses (from the first testing) and thereby proceeding to reproduce them on the second testing.

3. Testing the students before and after the Christmas holidays would curtail the amount of student discussion of the test and the learning of new mathematical skills, both of which would have an effect upon the second testing (and thus produce a spuriously low reliability coefficient).

Forty-five students participated in the first testing and fifty students wrote the retest. From these testings, only forty-one students had taken both the test and the retest. Therefore, the author had two scores for each of these forty-one students. Plotting these data on a scatter diagram provides a configuration that can, at a glance, reveal the level of reliability of the test. (Figure 5.1)

The points on the graph in Figure 5.1 are arranged in an elongated manner which indicates that the test is highly reliable. Also note that if a straight line was drawn to represent the plotted points, its extension would intersect the vertical axis slightly above the origin of the graph. This is an indication that the results

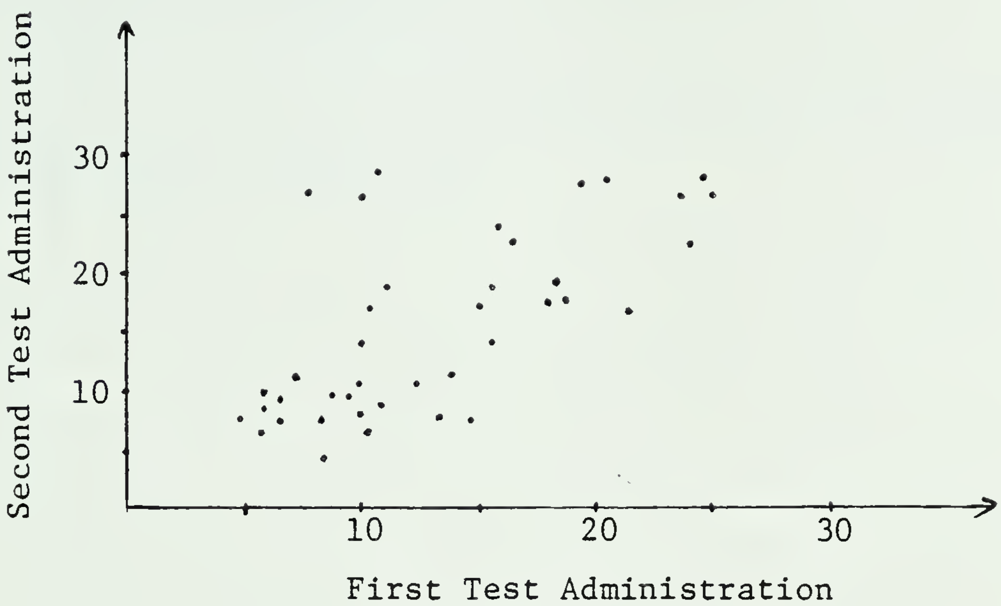


FIGURE 5.1 SCATTER DIAGRAM OF MATHEMATICS 20 PROBLEM-SOLVING TEST SCORES RESULTING FROM TWO TEST ADMINISTRATIONS

on the retest were generally slightly higher than those from the first test administration.

The Pearson product-moment coefficient of correlation is the most suitable means of representing the degree to which the points on the graph conform to a straight line. According to Ebel (1965: 313):

> The correlation between the set of scores obtained on the first administration of the test and that obtained on the second administration yields a test-retest reliability coefficient.

The test-retest reliability coefficient obtained for the Mathematics 20 Problem-Solving Test was 0.831.

CHAPTER VI

DESCRIPTION AND ANALYSIS OF RESULTS

The fundamental purpose of this thesis was to develop a unit for training students in a heuristic problem-solving process. The problem under consideration in this research study was to develop the above-mentioned unit and to determine whether this unit, after being taught to grade eleven (Mathematics 20) students, yielded any significant difference in problem-solving test scores when compared to a more conventional approach to problem solving. In the "Statement of the Problem" given in Chapter I, section B, the researcher delineated three questions which define the above-mentioned purpose. It is the aim of this chapter to describe and analyze the data with the intention of answering each of the three questions specified in Chapter I, section B. Prior to doing this, the author will address the question: Did the students of the two groups differ initially in terms of factors that were relevant to the study?

A. Students' Entrance Behavior

1. Descriptive Information for the Sample

(a) Overall Sample

The sample that was used in this research study was composed of seventeen males and nineteen females. Originally, the sample contained fifty-nine pupils, but eight students from the experimental group (as shown in Table 6.1) and fifteen students from the control group (as shown in Table 6.2) were absent from at least one test and

were therefore excluded from the sample.

NAME OF TEST	NUMBER OF ABSENTEES
Mathematics 20 Problem-Solving Test (Pretest)	3
Five-Method Comprehension Test	0
Heuristic Problem-Solving Test	3
Characteristics and Solution-Methods Test	5
Mathematics 20 Problem-Solving Test (Post-test)	2
Total Students Missing at Least One of the Above Tests	8

TABLE 6.1 STUDENT ABSENTEEISM IN THE EXPERIMENTAL GROUP

NAME OF TEST	NUMBER OF ABSENTEES
Mathematics 20 Problem-Solving Test (Pretest)	3
Heuristic Problem-Solving Test	9
Mathematics 20 Problem-Solving Test (Post-test)	8
Total Students Missing at Least One of the Above Tests	15

TABLE 6.2 STUDENT ABSENTEEISM IN THE CONTROL GROUP

The school enrollment was approximately two thousand students, the majority of whom came from working class parents, middle to lower-middle class socioeconomic status. Approximately eighteen percent of the students live in rural areas and are bussed in.

The philosophy of the school is similar to other Edmonton Public High Schools in the sense that it realizes individual differences and

seeks to provide for their needs by offering a diversified selection of courses and programs. According to the article entitled "Program Budget Goal Statements and Objects" (1976: 4):

> The Faculty acknowledges that social, physical, intellectual, cultural, and emotional differences exist among students. With the limitations of time, resources, teaching ability, subject expertise, the Faculty will provide an appropriate program to enhance the learning of each student.

There is an emphasis on preparation for a lifetime of continuous education through direct interaction with parents and community. Their educational philosophy statement says:

> Through direct involvement in the educational process . . . will attempt to provide an appropriate program that will enhance the preparation for a lifetime of continuous learning. Responsibility and accountability for the educational process is shared by teachers, students, and parents. (1976: 4)
>
> The community must be as much a part of the learning environment within the school, as the school is part of the community environment. (1976: 5)

The mean age of the thirty-six pupils, comprising the author's sample, was 16.8 years. Their mean intelligence quotient was based on the average of the verbal and the nonverbal scores of the Lorge Thorndike Intelligence Test. The mean of the full-scale scores was 118.7. The mean score on the Mathematics 10 course (final mark) was 70.6%. The concepts covered in the course were: algebra of polynomials, exponents, radicals, real numbers (and equations involving them), coordinate geometry, geometry, and trigonometry. The mid-term mark for Mathematics 20 students indicated an evaluation of the students' understanding of the following concepts: radicals, relations, quadratic functions, and quadratic equations and

inequations. The mean score was 68.8%. A final descriptive statistic of the overall sample is the mean score on the Mathematics 20 Problem-Solving Test (pretest). The prerequisite skills necessary for obtaining a high score on this test were mostly process skills; however the students were required to know the basic arithmetic operations, Pythagoras' Theorem, and systems of linear equations. The mean score on this test was 34.4%.

(b) Experimental Group

The experimental treatment group was composed of nine males and ten females, whose average age was 16.9 years. The mean of the full-scale scores on the Lorge Thorndike Intelligence Test was 120.7. The respective mean scores on the Mathematics 10 course (final mark), the Mathematics 20 mid-term mark, and the Mathematics 20 Problem-Solving Test (pretest) were 71.2%, 64.5%, and 34.9%.

(c) Control Group

The control treatment group was composed of eight males and nine females, whose average age was 16.6 years. The mean of the full-scale scores on the Lorge Thorndike was 116.9. The respective mean scores on the Mathematics 10 course (final mark), the Mathematics 20 mid-term mark, and the Mathematics 20 Problem-Solving Test (pretest) were 69.9%, 73.5%, and 33.8%. Table 6.3 provides a contrast of all the descriptive statistics for the overall sample, the experimental group, and the control group.

CHARACTERISTICS		OVERALL SAMPLE	EXPERIMENTAL GROUP	CONTROL GROUP
Socioeconomic Background		Middle and Lower-Middle Class		
Number of Males and Females		19 females 17 males	10 females 9 males	9 females 8 males
Mean Age of Students		16.8 years	16.9 years	16.6 years
Intelligence	Mean	118.7	120.7	116.8
	Standard Deviation	9.6	9.8	9.4
Mathematics 10 Final Mark	Mean S.D.	70.6% 11.1	71.2% 10.9	69.9% 12.0
Mathematics 20 Mid-Term Mark	Mean S.D.	68.8% 14.5	64.5% 15.4	73.5% 12.4
Mathematics 20 Problem-Solving Test (Pretest)	Mean S.D.	34.4% 20.1	34.9% 18.4	33.8% 23.0

TABLE 6.3 DESCRIPTIVE INFORMATION FOR THE OVERALL SAMPLE AND ITS TWO COMPONENT PARTS

2. Analysis and Discussion of the Entrance Behavior of the Two Treatment Groups

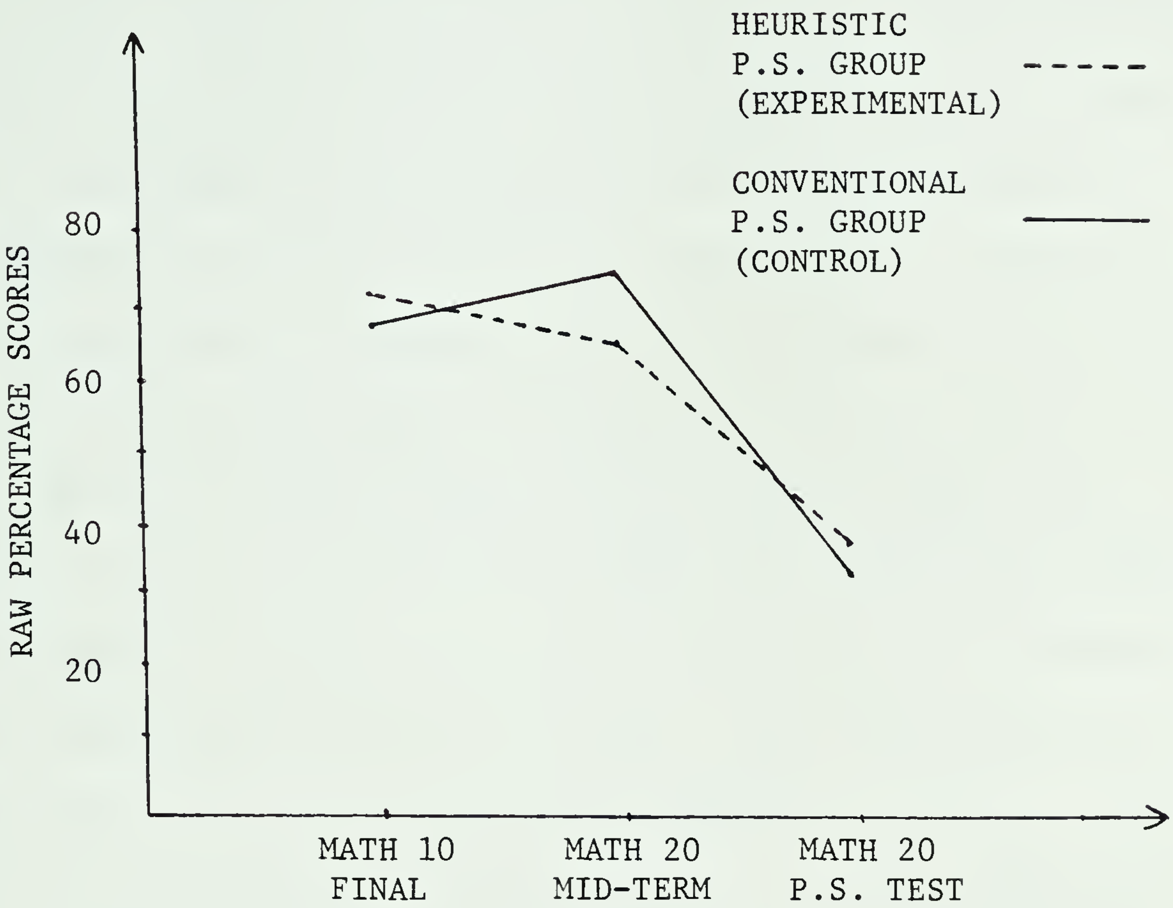


FIGURE 6.1 PROFILE OF PRETEST MEANS FOR THE HEURISTIC PROBLEM-SOLVING TREATMENT GROUP AND THE CONVENTIONAL PROBLEM-SOLVING TREATMENT GROUP

The appropriate results necessary to draw the graph, Figure 6.1, are given in Table 6.3. These results indicate that there is a small difference between the means of the two groups on the Mathematics 10 final mark (71.2% - 69.9% = 1.3%), a larger difference on the Mathematics 20 mid-term mark (73.5% - 64.5% = 9.0%), and a small difference on the Mathematics 20 Problem-Solving Test (34.9% - 33.8% = 1.1%).

The following strategy was adopted as a means of initially equating the two groups. The Mathematics 20 Problem-Solving Test, administered as a pretest to both groups on May 2, 1977, was utilized

as the equating instrument. The thirty-six scores from this pretest were ranked from the top mark along a descending continuum to the lowest mark. Then the scores were separated into three equal groups of twelve; top, middle and low. The purpose of defining the three levels, top, medium and low was to provide an accurate means of determining the equality of the two treatment groups. Table 6.4 shows the distribution of people into their different categories on the basis of the above-mentioned strategy. It should be noted that equal numbers of students (six from each treatment group) were located in both the "low" and the "middle" levels and approximately equal numbers of students (five and seven from the two treatment groups) were located in the "top" group. Table 6.5 reveals the initial mean scores for each of these categories, as well as the overall treatment means for the two groups. Although levels were used as a variable in testing the data, the reporting on between level differences is not of primary importance in this study.

	NUMBER OF STUDENTS			
GROUP	LOWER ONE-THIRD	MIDDLE ONE-THIRD	UPPER ONE-THIRD	TOTAL STUDENTS
Experimental Group	6	6	7	19
Control Group	6	6	5	17
Total Students	12	12	12	36

TABLE 6.4 NUMBER OF STUDENTS IN EACH LEVEL BASED ON THE MATHEMATICS 20 PROBLEM-SOLVING TEST (PRETEST)

GROUP	MEAN SCORE OF STUDENTS			
	LOWER ONE-THIRD	MIDDLE ONE-THIRD	UPPER ONE-THIRD	GROUP MEAN
Experimental Group	15.8	32.8	53.0	34.9
Control Group	11.7	31.3	63.4	33.8
Group Means	13.8	32.1	57.3	34.4

TABLE 6.5 MEAN SCORES OF STUDENTS IN EACH LEVEL BASED ON MATHEMATICS 20 PROBLEM-SOLVING TEST (PRETEST)

B. Reaction to the Treatments

1. Students' Reaction to the Treatment

In this subsection, the students' responses to each of the six questions will be analyzed and then representative responses of both treatment groups will be selected. A complete listing of all the experimental and control groups' responses is located in Appendices I and J respectively.

(a) Question One

"Did you find that the procedure presented (e.g. Unknown, Given, etc.) was useful in solving problems? Why or why not?"

(i) Quantitative Responses

STUDENT RESPONSES	EXPERIMENTAL GROUP'S RESPONSES EXPRESSED AS PERCENTAGES	CONTROL GROUP'S RESPONSES EXPRESSED AS PERCENTAGES
Yes	87%	59%
Maybe	0%	23%
No	13%	18%

TABLE 6.6 QUANTITATIVE STUDENT RESPONSES TO QUESTION ONE (EXPERIMENTAL AND CONTROL GROUPS)

(ii) Qualitative Responses

Two typical responses by the experimental group were:

"I feel it was helpful because going through a problem step by step really helps you get to know how to do it."

"It helped us organize our thoughts and procedures in problem solving."

Two typical responses by the control group were:

"Yes—it gave different ideas for solving different basic types of problems."

"I found it easier to do some problems than others."

(b) Question Two

"What method of solution did you feel most comfortable with?"

SOLUTION METHOD CHOSEN	EXPERIMENTAL GROUP'S CHOICE AS A PERCENTAGE	CONTROL GROUP'S CHOICE AS A PERCENTAGE
Decomposing and Recombining (Type #5)	37%	56%
Generalization (Type #1)	26%	28%
Working Backward (Type #2)	18%	8%
Analogy (Type #4)	15%	4%
Specialization (Type #3)	4%	4%

TABLE 6.7 QUANTITATIVE STUDENT RESPONSES TO QUESTION TWO (EXPERIMENTAL AND CONTROL GROUPS)

(c) Question Three

"What method of solution did you feel least comfortable with?"

SOLUTION METHOD CHOSEN	EXPERIMENTAL GROUP'S CHOICE AS A PERCENTAGE	CONTROL GROUP'S CHOICE AS A PERCENTAGE
Specialization (Type #3)	61%	63%
Analogy (Type #4)	23%	22%
Working Backward (Type #2)	8%	4%
Generalization (Type #1)	8%	4%
Decomposing and Recomposing (Type #5)	0%	7%

TABLE 6.8 QUANTITATIVE STUDENT RESPONSES TO QUESTION THREE (EXPERIMENTAL AND CONTROL GROUPS)

(d) Question Four

"Do you feel that you are a better problem solver now than before? Why or why not?"

(i) Quantitative Responses

STUDENT RESPONSES	EXPERIMENTAL GROUP'S RESPONSES EXPRESSED AS PERCENTAGES	CONTROL GROUP'S RESPONSES EXPRESSED AS PERCENTAGES
Yes	78%	50%
Maybe	13%	9%
No	9%	41%

TABLE 6.9 QUANTITATIVE STUDENT RESPONSES TO QUESTION FOUR (EXPERIMENTAL AND CONTROL GROUPS)

(ii) Qualitative Responses

Two typical responses by the experimental group were:

"Yes I do believe I have improved because now I have some idea of how to go about solving a problem; it is no longer hit and miss."

"Yes, because now I am able to look at a problem and evaluate it without getting too frustrated or discouraged."

Two typical responses by the control group were:

"Yes, because I have learned many ways to solve a problem."

"No, because not enough practice was given in each of the different types of problems. I didn't catch on."

(e) Question Five

"Do you think this unit will help you in solving new problems in Mathematics 20? If yes, how? If no, why not?"

(i) Quantitative Responses

STUDENT RESPONSES	EXPERIMENTAL GROUP'S RESPONSES EXPRESSED AS PERCENTAGES	CONTROL GROUP'S RESPONSES EXPRESSED AS PERCENTAGES
Yes	78%	64%
Maybe	18%	18%
No	4%	18%

TABLE 6.10 QUANTITATIVE STUDENT RESPONSES TO QUESTION FIVE (EXPERIMENTAL AND CONTROL GROUPS)

(ii) Qualitative Responses

Two typical responses by the experimental group were:

"Yes, I think they will help me solve new problems in Math 20 in that there is now a set of steps for me to follow."

"Yes, the unit has shown me new methods of solution and has given me practice in decomposing and recombining."

Two typical responses by the control group were:

"Yes, because I can relate the problems (in Math 20) to one of the five types and be able to find the solution easier."

"Yes. I've got the experience."

(f) Question Six

"Other comments and concerns."

Two typical responses by the experimental group were:

"The unit was easy to follow and understand."

> "I think the course was all in all a good one, but more time should be spent on specialization and analogy and less on generalization and decomposing and recombining."

Two typical responses by the control group were:

> "If more practice would have been given in different types of problem solving, I would have done a lot better on the tests and in the future."

> "I feel that special solutions should be more clear."

2. Teacher's Reaction to the Treatment

The cooperating teachers from both treatment groups were given the Teacher Opinion Survey on Thursday, May 26, 1977 and were asked to complete it at their convenience. Each cooperating teacher was asked to respond to the one treatment that they had observed. In this section the author will restate each of the five questions contained in the survey and subsequently document their responses. For simplicity, the cooperating teacher for the experimental group will be labelled CEG and the cooperating teacher for the control group will be designated CCG.

(a) Question One

> "As an experienced teacher yourself, do you feel that other Mathematics 20 teachers would want to make use of this method of problem solving?"

CEG: "Yes, with some modification which would make the method more streamlined."

CCG: "Yes, I believe they would."

(b) Question Two

> "Do you feel that the students benefited by learning to use this method of problem solving?"

CEG: "Very definitely."

CCG: "The benefit was both for the course and for solving problems outside of math."

(c) Question Three

"What did you like, in particular, about this method of problem solving?"

CEG: "The fact that the students had a model to follow and that the students had an approach to the problem; e.g. What are the characteristics?"

CCG: "Variety in approach; emphasis on logic; variety of problems presented."

(d) Question Four

"What did you dislike (if anything) about this method of problem solving?"

CEG: "It should be made clear that this was a mental model which at some time is simplified to the basic processes."

CCG: "The time taken from the regular course of studies."

(e) Question Five

"Please make other comments that you wish to."

CEG: "A good unit!"

CCG: "I feel that the gain to the student may well have offset the loss in instructional time for the regular course."

C. Analysis of Results Based upon Research Questions and Hypotheses

Three questions were raised in Chapter I, page 7. In this section, each of these questions will be examined and the data, both experimental and empirical, will be analyzed to ascertain the answer to each of the three questions.

1. Question One

Statement: "What attitudes do students and teacher have toward a heuristic problem solving instructional strategy?"

Scientific Hypothesis #1: "The students and the teacher will have a positive attitude toward a heuristic problem-solving instructional strategy."

In Chapter III, page 53, a set of criteria was established as a means of answering scientific hypothesis #1. The criterion was concerned with attitude toward the unit and sought an answer to the question: "Was there a positive reaction to the unit by both the students and the cooperating teacher?" Table 6.6, page 112, reveals that 87% of the students felt that the procedure presented in the unit was helpful in solving problems. Table 6.9, page 114, showed that 78% of the students felt that they were better problem solvers as a result of the unit. Table 6.10, page 115, indicated that 78% of the group thought that the unit would help in solving new problems in Mathematics 20. Generally, the students felt that it was a "very helpful course; interesting" (page 376). The cooperating teacher felt "very definitely" (page 116) that the students had benefited by this unit. He expressed his own reaction by stating that it was "a good unit!" (page 117).

2. <u>Question Two</u>

Statement: "Does such an instructional strategy affect the students' ability in solving problems peculiar to those utilized in that unit?"

Scientific Hypothesis #1: "It would seem probable that students who have been subjected to the heuristic problem-solving treatment, will successfully use given methods of solution in solving problems."

In Chapter III, page 54, a set of two criteria was established as a means of answering scientific hypothesis #2. The first criterion

was concerned with the Five-Method Comprehension Test and stated: "The researcher established an individual mean of 70% or above as an indication of successful use of a given method of solution in solving the problem." The second criterion was concerned with the Heuristic Problem-Solving Test and stated: "The researcher established an individual mean of 40% or above as an indication of successful use of a given method of solution in solving the problem."

The Five-Method Comprehension Test was administered in lesson ten, to the experimental group. Its purpose was to provide an indication of the students' mastery of the five different methods of solution. In lessons two through nine, the pupils were taught to use these five solution methods in solving designated problems; it was therefore necessary to ascertain the students' ability to solve problems whose solution methods were specified before presenting them with problems with undesignated solution methods.

From Table 6.11 it can be seen that the five individual means from the pilot study on the Five-Method Comprehension Test were 84%, 70%, 64%, 62%, and 88%, with a grand mean of 73.5%; while the respective five individual means from the main study on the Five-Method Comprehension Test were 77%, 70%, 42%, 63%, and 81%, with a grand mean of 66.5%.

The five problems contained in this test were taken directly from the section of the treatment that the student had already completed. Since this was a comprehension test, the five problems were altered so that the students were not able to simply memorize the problem solutions from lessons two to nine, and reproduce them on the test. This observation is noted to help explain the mean differences in the two sets of scores.

The author established an individual mean of 70% or above as an

EXPERIMENTAL GROUP	MEAN SCORES EXPRESSED AS PERCENTAGES					
	PROBLEM #1	PROBLEM #2	PROBLEM #3	PROBLEM #4	PROBLEM #5	
	GENERALIZATION	WORKING BACKWARD	SPECIALIZATION	ANALOGY	DECOMPOSING AND RECOMBINING	GRAND MEAN
Main Study	77	70	42	63	81	Mean = 66.5 S.D. = 15.0
Pilot Study	84	70	64	62	88	Mean = 73.5

TABLE 6.11 INDIVIDUAL MEANS AND GRAND MEAN FOR THE FIVE-METHOD COMPREHENSION TEST

indication of successful use of a given method of solution in solving the problem. In the pilot study, the individual means of 64% (specialization-type problem) and 62% (analogy-type problem) were evidence of insufficient comprehension of these two solution methods. Although part of lesson eleven, the lesson following the Five-Method Comprehension Test, was utilized to clarify and re-emphasize the strategy employed in specialization and analogy, the results of the Heuristic Problem-Solving Test continued to reveal the students' misunderstanding of specialization as a method of solution (individual mean of 27%). At the conclusion of the pilot study the author felt that this low mean could be significantly raised if further clarification were given to the appropriate section in the experimental treatment; that is, hints were to be provided in those problems involving specialization as their solution method.

This provision did not enlighten the experimental treatment group in the main study as evidenced by the individual mean of 42% (specialization-type problem) on the Five-Method Comprehension Test. The Heuristic Problem-Solving Test, when administered to the heuristic problem-solving treatment group, further revealed the weakness of the students' usage of specialization as a solution method. The individual mean for specialization was 25%.

Therefore, in response to the first specific question: The methods of solution, generalization, working backward, and decomposing and recombining were successfully used in solving problems. Analogy and specialization were utilized less than successfully by the pupils in solving problems.

Scientific Hypothesis #3: "It would seem probable that students, who have been subjected to the heuristic

> problem-solving treatment, will perform better on a heuristic problem-solving test, than students who have been taught using a conventional problem-solving treatment."

Both groups received training on the same set of problems. The experimental group, however, was taught the general steps of the heuristic problem-solving process (explained in Chapter II, section A), as well as how to characterize problems and to associate them with their appropriate solution methods (Appendix G, lessons 2-9, 11-16); whereas the control group received no such training. Therefore, the experimental treatment group had the advantage of having been explicitly taught to develop concepts, that is, the five solution methods. According to Davis (1973: 38):

> Problem solving should become easier whenever an unfamiliar problem can be identified as a member of a class of problems whose solution-strategy is known.

Null hypothesis: There is no significant difference in mean scores between the instructional methods as measured by the Heuristic Problem-Solving Test.

The grand means and the individual problem means for the experimental and control groups are given in Table 6.12.

These data were placed into three categories as shown in Table 6.13 and Figure 6.2.

The interaction effect was not significant at the .05 level; therefore the author examined the overall main effects and not the simple main effects. Table 6.14 shows the computed F-statistic, 0.04, and the critical F-statistic, 4.17 ($\alpha = .05$). Therefore, in response to the scientific hypothesis #2, the treatment effect is not significant and the null hypothesis cannot be rejected.

GROUP	PROBLEM #1 DECOMPOSING AND RECOMBINING	PROBLEM #2 SPECIALIZATION	PROBLEM #3 WORKING BACKWARD	PROBLEM #4 ANALOGY	PROBLEM #5 GENERALIZATION	GROUP MEAN
	MEAN SCORES EXPRESSED AS PERCENTAGES					
Experimental Group	57	25	78	54	45	Mean 51.9 S.D. 20.5
Control Group	53	17	88	48	42	Mean 49.7 S.D. 20.1
Overall Sample						Mean 50.9

TABLE 6.12 INDIVIDUAL PROBLEM MEANS AND GRAND MEAN FOR THE HEURISTIC PROBLEM-SOLVING TEST

GROUP	LOWER ONE-THIRD NO. STUD.	MEAN	MIDDLE ONE-THIRD NO. STUD.	MEAN	UPPER ONE-THIRD NO. STUD.	MEAN	TREATMENT GROUP MEANS
Experimental Group	6	40.3	6	56.7	7	57.7	51.9
Control Group	6	46.3	6	44.5	5	60.0	49.7
Group Level Mean		43.3		50.6		58.7	50.9

TABLE 6.13 INDIVIDUAL GROUP MEANS AND GRAND MEAN FOR THE HEURISTIC PROBLEM-SOLVING TEST

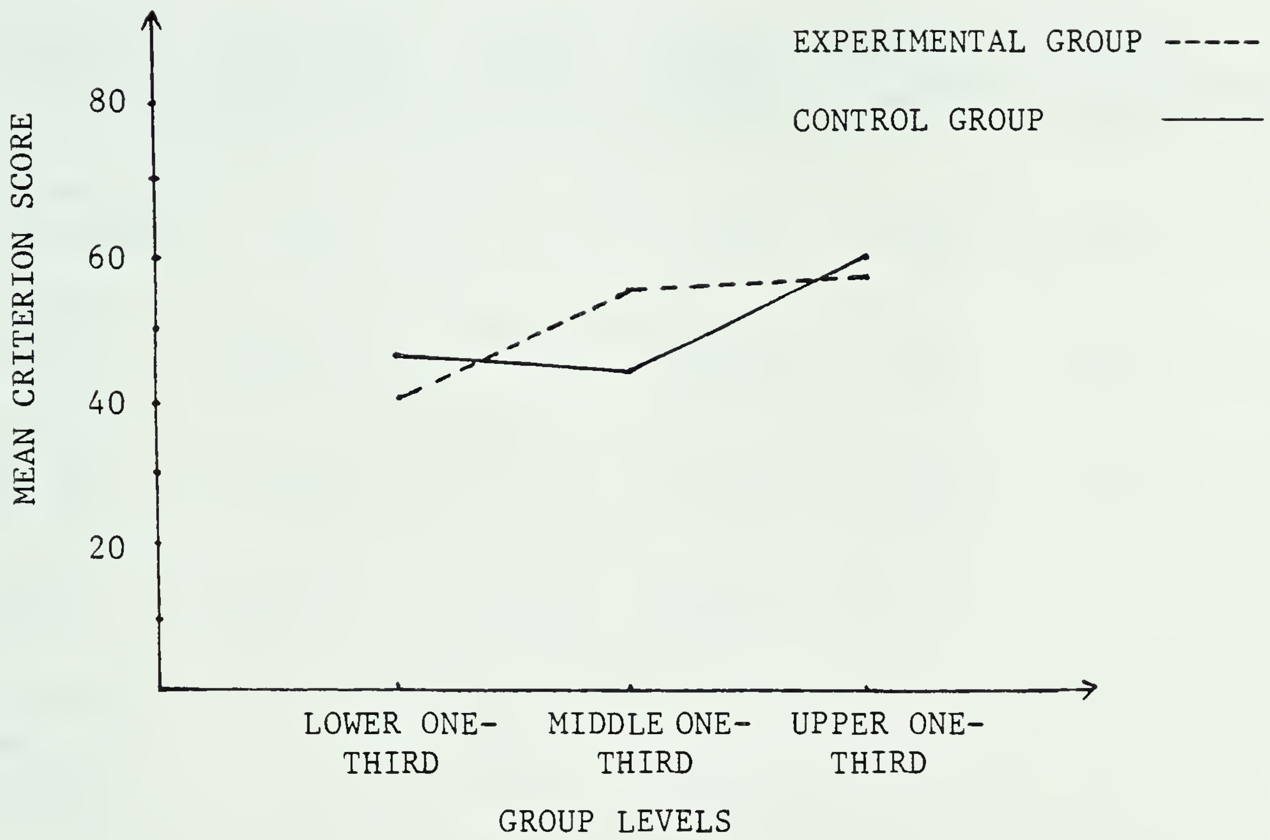


FIGURE 6.2 PROFILES OF SIMPLE MAIN EFFECTS FOR THREE GROUP LEVELS AND TWO TREATMENTS ON THE HEURISTIC PROBLEM-SOLVING TEST

SOURCE OF VARIATION	DEVIATION SUM SQUARES	DEGREES FREEDOM	VARIANCE ESTIMATE	OBSERVED F RATIO	CRITICAL F RATIO ($\alpha = .05$)
Between Treatments	2.54	1	2.54	0.04	4.17
Between Levels	242.10	2	121.05	1.79	3.32
Inter-action Effect	92.53	2	46.27	0.68	3.32
Within Cells	12066.94	30	402.23		
Adjusted Mean Square For Error			67.72		

TABLE 6.14 TWO-WAY ANALYSIS OF VARIANCE UTILIZING THE HEURISTIC PROBLEM-SOLVING TEST

Scientific Hypothesis #4: "It would seem probable that there is a strong relationship between scores on a heuristic problem-solving test and scores on a characteristics and solution-methods test, for students who have been subjected to the heuristic problem-solving treatment."

In Chapter III, page 58, a null hypothesis was established as a means of answering scientific hypothesis #4. It stated:

> The correlation coefficient between the scores on the Heuristic Problem-Solving Test and the Characteristics and Solution Methods Test is zero.

The scoring procedure that was used in marking the students' answers to the Heuristic Problem-Solving Test is located in Appendix G (lesson seventeen) of this thesis. Each of the five problems was allotted six marks; no marks were given for naming the characteristics and the associated solution method, but rather the six points were distributed throughout the solution, beginning at a place after the identification of the method of solution to be utilized in each problem. The results of the Heuristic Problem-Solving Test are located in Table 6.12.

The scoring procedure that was used in marking the students' answers to the Characteristics and Solution-Methods Test is located in Appendix G (lesson eighteen) of this thesis. Each of the ten problems was allotted two marks: one mark for naming the characteristics and the second mark for naming the correct solution method. The results of the Characteristics and Solution-Methods Test and the Heuristic Problem-Solving Test are located in Table 6.15.

TEST	WORKING BACK-WARD	DECOM-POSING AND RECOM-BINING	ANALOGY	GENERALIZA-TION	SPECIALI-ZATION	TEST MEAN
	MEAN SCORES EXPRESSED AS PERCENTAGES					
	NAME OF PROBLEM TYPE					
Heuristic P.S. Test	78	57	54	45	25	51.9
Character-istics and Solution Methods Test	84	84	97	51	80	79.6

TABLE 6.15 INDIVIDUAL MEAN SCORES AND GRAND MEANS FOR THE HEURISTIC PROBLEM-SOLVING TEST AND THE CHARACTERISTICS AND SOLUTION-METHODS TEST

Although the author's primary concern is with the relationship between the Heuristic Problem-Solving Test and the Characteristics and Solution-Methods Test, he has documented the six correlations that exist among the four tests in Table 6.16.

The Pearson product-moment correlation coefficient was used to test the relationships between pairs of tests because the scoring procedure used in evaluating the students' answers in each of the four tests was continuous. According to Ferguson (1976: 102):

> The most widely used measure of correlation is the Pearson product-moment correlation coefficient. This measure is used where the variables are quantitative, that is, of the interval or ratio type.

For each of the six correlation coefficients, there is an

	HEURISTIC PROBLEM-SOLVING TEST	CHARACTER-ISTICS AND SOLUTION-METHODS TEST	FIVE-METHOD COMPRE-HENSION TEST	MATH 20 PROBLEM-SOLVING TEST
Heuristic P.S. Test	1.00	0.58	0.21	0.57
Character-istics and Solution-Methods Test		1.00	0.65	0.61
Five-Method Comprehension Test			1.00	0.56
Math 20 P.S. Test				1.00

TABLE 6.16 PEARSON PRODUCT-MOMENT CORRELATION COEFFICIENTS FOR THE SIX DIFFERENT COMBINATIONS OF THE FOUR TESTS

associated confidence interval, that is, a range of possible coefficients, any one of which could have been the representative one given in Table 6.7. According to Winer (1971: 10):

> An interval estimate is frequently referred to as a confidence interval for a parameter. The two extreme points in this interval, the upper and lower confidence bounds, define a range of values within which there is a specified likelihood (or level of confidence) that the parameter will fall.

Walker and Lev (1953: 476) provide the reader with a chart showing the confidence belt for the correlation coefficient p when $\alpha = 0.05$. If the sample size is nineteen, then each of six correlation coefficients have the following confidence intervals:

PEARSON PRODUCT-MOMENT CORRELATION COEFFICIENT	LOWER CONFIDENCE BOUNDARY	UPPER CONFIDENCE BOUNDARY
0.21	-0.26	0.59
0.56	0.14	0.80
0.57	0.16	0.80
0.58	0.17	0.81
0.61	0.23	0.82
0.65	0.29	0.84

TABLE 6.17 UPPER AND LOWER CONFIDENCE BOUNDS FOR EACH PEARSON PRODUCT-MOMENT CORRELATION COEFFICIENT

Table 6.16 contains the Pearson product-moment correlation coefficient for the Heuristic Problem-Solving Test with the Characteristics and Solution-Methods Test. It is 0.58. The confidence bounds for this coefficient are 0.17 and 0.81. To reject the null hypothesis

for scientific hypothesis #4, at the .05 level of significance, a correlation coefficient of at least 0.45 is necessary (Hays, 1963: 531). Therefore, in response to scientific hypothesis #4, the correlation coefficient 0.58 reveals that there is a significant relationship between the two tests, the Heuristic Problem-Solving Test and the Characteristics and Solution-Methods Test. The coefficient indicates that the heuristic problem-solving treatment group utilized problem characterization and the association of characteristics and solution method in solving the problems contained in the Heuristic Problem-Solving Test.

The empirical and experimental evidence gathered to answer the three scientific hypotheses which comprise Question Two revealed that the heuristic problem-solving strategy does not significantly affect the students' ability in solving problems peculiar to those utilized in that strategy. It was found that the heuristic problem-solving treatment group: (1) did not score significantly better on a heuristic problem-solving test than students who had been trained to use a conventional problem-solving treatment; (2) was not successful in utilizing the solution methods, specialization and analogy; and (3) utilized problem characterization and the association of characteristics and solution method in solving the problems contained in the Heuristic Problem-Solving Test.

3. Question Three

Statement: "Does such an instructional strategy affect the students' ability in solving those problems which are peculiar to those utilized at a specific grade and course level?"

Scientific Hypothesis #5: "It would seem probable that students, who have been subjected to the heuristic problem-solving treatment, will perform better on a Mathematics 20 problem-

> solving test, than students who have been taught using a conventional problem-solving treatment."

Lucas' (1974) study was performed on first-year university students; his results indicating that students who were trained in heuristic processes obtained more accurate results than those who were not. Ashton's (1962) study revealed similar results, but was performed on grade IX mathematics students. It was hoped that further support for this hypothesis would result from the heuristic problem-solving treatment located in Appendix G of this thesis.

Null Hypothesis: There is no significant difference in mean scores between the two instructional methods as measured by the Mathematics 20 Problem-Solving Test.

The results of the pretest and the post-test for the experimental and control groups are presented in Table 6.18.

		MEAN SCORES EXPRESSED AS PERCENTAGES					GRAND MEANS & STANDARD DEVS.	
GROUP	TEST	PROBLEM #1	PROBLEM #2	PROBLEM #3	PROBLEM #4	PROBLEM #5		
Experi-mental Group	Pretest	26	30	25	33	60	Mean	34.9
							S.D.	18.4
	Post-test	51	74	37	32	51	Mean	49.0
							S.D.	25.4
Control Group	Pretest	15	52	28	30	44	Mean	33.8
							S.D.	23.0
	Post-test	42	57	27	39	50	Mean	43.0
							S.D.	24.4

TABLE 6.18 INDIVIDUAL PROBLEM MEANS, GRAND MEANS, AND THEIR RESPECTIVE STANDARD DEVIATIONS FOR THE MATHEMATICS 20 PROBLEM-SOLVING TEST

In Chapter VI, page 109, a detailed outline of the method that the researcher utilized in order to initially equate the two groups

is given. Analyzing the data from the Mathematics 20 Problem-Solving Test (post-test) on that basis, the researcher initially placed the data into their proper categories as shown in Table 6.19 and as viewed more clearly in Figure 6.3.

	LOWER ONE-THIRD		MIDDLE ONE-THIRD		UPPER ONE-THIRD		
GROUP	NO STUD.	MEAN	NO STUD.	MEAN	NO STUD.	MEAN	TREATMENT GROUP MEANS
Experi-mental Group	6	23.5	6	51.2	7	69.0	49.0
Control Group	6	23.0	6	38.2	5	72.8	43.0
Group Level Mean		23.3		44.7		70.6	46.2

TABLE 6.19 INDIVIDUAL GROUP MEANS AND GRAND MEAN FOR THE MATHEMATICS 20 PROBLEM-SOLVING TEST

The interaction effect was not significant at the .25 level, therefore the author examined the overall main effects and not the simple main effects. Table 6.20 shows the computed F-statistic, 0.37, and the critical F-statistic, 1.38 ($\alpha = .25$). Therefore, in response to scientific hypothesis #5, the treatment effect is not significant and the null hypothesis cannot be rejected. It should be noted from Table 6.20 that there is a significant group level effect ($\alpha = .01$).

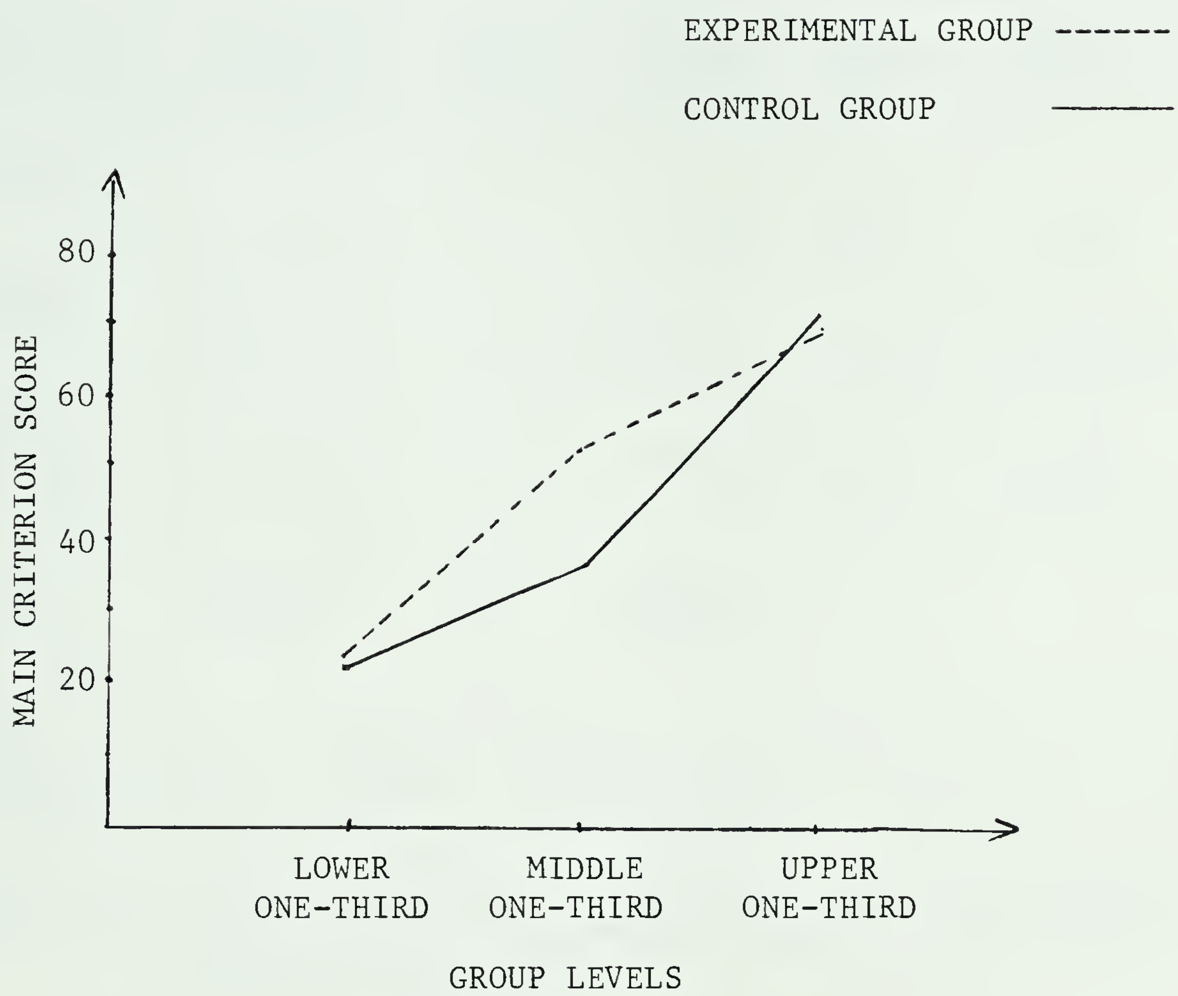


FIGURE 6.3 PROFILES OF SIMPLE MAIN EFFECTS FOR THREE GROUP LEVELS AND TWO TREATMENTS ON THE MATHEMATICS 20 PROBLEM-SOLVING TEST (POST-TEST)

SOURCE OF VARIATION	DEVIATION SUM SQUARES	DEGREES FREEDOM	VARIANCE ESTIMATE	OBSERVED F RATIO	CRITICAL F RATIO (α = .25)
Between Treatments	15.68	1	15.68	0.37	1.38
Between Levels	2278.04	2	1139.02	27.20	5.39 (α = .01)
Interaction Effect	76.17	2	38.08	0.91	1.45
Within Cells	7461.97	30	248.77		
Adjusted Mean Square For Error			41.87		

TABLE 6.20 TWO-WAY ANALYSIS OF VARIANCE UTILIZING THE MATHEMATICS 20 PROBLEM-SOLVING TEST

Scientific Hypothesis #6: "It would seem probable that students, who have been subjected to the heuristic problem-solving treatment, will perform better on the post-test of a Mathematics 20 problem-solving test, than they did on its pretest."

Null Hypothesis: There is no significant difference in mean scores between the pretest and post-test as measured by the Mathematics 20 Problem-Solving Test.

The results contained in Table 6.21 reveal that there is a significant difference between pretest and post-test scores at the 0.01 level. Therefore, it can be concluded that the experimental treatment group made a dramatic improvement in their problem-solving score as measured by the Mathematics 20 Problem-Solving Test. The experimental group's pretest mean was 34.9% and their post-test mean was 49.0%.

SOURCE OF VARIATION	DEVIATION SUM SQUARES	DEGREES FREEDOM	VARIANCE ESTIMATE	OBSERVED F RATIO	CRITICAL F RATIO (α = .01)
Between Pretest and Post-test	1890.10	1	1890.10	19.36**	8.28
Within Cells	1756.90	18	97.61		

TABLE 6.21 ONE-WAY ANALYSIS OF VARIANCE WITH REPEATED MEASUREMENTS UTILIZING THE MATHEMATICS 20 PROBLEM-SOLVING TEST

The experimental data gathered to answer the two scientific hypotheses which comprise Question Three revealed that the heuristic problem-solving group:

1. did not differ significantly from a conventional problem-

solving group as measured by the students' ability in solving problems which are peculiar to those utilized at a particular course and grade level;

2. made a dramatic improvement from pretest to post-test as measured by the students' ability in solving those problems which are peculiar to those utilized at a particular grade and course level. The conventional problem-solving group made a similar improvement.

CHAPTER VII

DISCUSSION OF RESULTS, SUMMARY, AND IMPLICATIONS

A. Discussion of Results

The purpose of the present study was to develop a unit for training students in a heuristic problem-solving process. The major thrust of this unit was to instruct pupils in the transfer process. That is, students were trained to characterize a problem, to associate the characteristics with a method of solution, and to use this solution method to solve the problem.

In this thesis, the author has developed and implemented a heuristic problem-solving unit of instruction based on Polya's model at the grade eleven level. Unlike either Crutchfield's (1967) or Wilson's (1967) self-instructional mode, the author's treatment utilized direct teacher involvement. The author's study was not a clinical investigation as was Kantowski's nor did it confine itself to a specific subject matter domain.

Progressing from the limitations of Ashton's (1962) study and based upon the recommendations of Kilpatrick (1967, 1969), Henderson and Pingry (1953), Wilson (1967) and Gurova (1968) the author developed a general heuristic framework for solving problems; incorporating within this approach, five methods of solution. This instructional mode is contained within a problem-solving unit which has been designed to be taught as a separate unit in the Mathematics 20 curriculum. Henderson and Pingry (1953), Gurova (1969) and Kantowski (1977) placed

a heavy emphasis on the teaching of process in problem solving; in response to their convictions, the author has developed a unit for training students in a heuristic problem-solving process.

In the "Statement of the Problem" given in Chapter I, pages 6-7, the author delineated three questions which defined this study's purpose. It is the aim of this chapter to discuss the results of Chapter VI with the intention of interpreting these results and stating the implications of this study.

Question One was concerned with the attitude of both students and teacher to the heuristic problem-solving unit that was both developed and implemented by the author. The empirical data gathered to answer Question One revealed that this unit was received favorably by both the students and the cooperating teacher. The results of Crutchfield's (1965) study were similar to the author's findings. He reported some highly successful and significant work he had done in teaching children to apply heuristic strategies to non-mathematical problems. Crutchfield had remarkable success in improving attitudes toward problem solving.

Questions Two and Three dealt with the <u>evaluation</u> of this unit. The results revealed that the heuristic problem-solving treatment group did not perform better than the control group on either of the two problem-solving tests. One test contained problems directly related to the unit and the other test contained problems directly related to the Mathematics 20 course of studies.

The <u>conclusion</u> that there was no significant treatment difference, in either case, provides evidence to support the assertion that

the two important variables in the heuristic problem-solving treatment which were excluded from the conventional problem-solving treatment, had little or no effect in facilitating improved problem-solving scores. The two variables are: the general heuristic approach to problem solving; the characterization of problems and their association with appropriate solution methods.

The heuristic problem-solving unit was based on the theory of "Going Beyond the Information Given" (pages 27-29) and the "Transfer of General Principles" (pages 30, 31). According to the former theory, when a problem solver is confronted with a problem, he wants to be able to use his knowledge gained as a result of past experience in solving problems to solve the one presently before him. It is therefore important that the individual has mentally organized his experiences into appropriate categories, each with its own defining properties. Once a problem has been placed in a given category, on the basis of its containing some of the basic defining properties of that class, then by inference, the problem is given class identity. Having bestowed class identity upon a given problem, a further inference is made; that is, all of the properties of the class can be attributed to the given problem and used in its solution. To illustrate this theory specifically in terms of the heuristic problem-solving unit, consider the following example. An individual is attempting to solve a problem. He identifies certain characteristics of the problem; for example, that the outcome of the problem is given and that it has a trial-and-error beginning. Based upon these properties, he infers that the problem can be placed into the category

labelled 'working backward.' Furthermore, the problem can be solved using the method of solution 'working backward.'

The author hypothesized, with support from Davis (1973) and Hendrickson and Schroeder (1941), that this treatment group would perform better than the conventional problem-solving group on problem-solving tests. One explanation for the lack of fit between the results of research studies gathered to support the hypothesis and the author's research findings is that both treatment groups actually characterized the problems. Although the control group was not explicitly taught problem characterization, they did implicitly characterize problems. For example, the experimental group was taught the following two characteristics for problems using working backward as their solution method:

1. The problem had a trial-and-error beginning;
2. The outcome of the problem is known.

The control group, on the other hand, verbalized that this type of problem was solved by working backward from the end to the beginning of the problem.

In summary, the failure to exclude the variable 'problem characterization and association with appropriate solution method' from the control treatment may have been the reason for the nonsignificant treatment difference on the two problem-solving tests. However, it should be noted that the heuristic problem-solving treatment group made a dramatic improvement from pretest to post-test as measured by the students' performance in solving those problems which were peculiar to those utilized at a particular grade and course level. Although

the control group made a similar improvement, the results from the Student Opinion Survey and Cooperating Teacher Opinion Survey revealed a more positive attitude toward the Heuristic Problem-Solving unit.

The major conclusion of this study is that a problem-solving unit, either the Heuristic Problem-Solving unit or the Conventional Problem-Solving unit, will produce a marked improvement in the students' problem-solving scores.

B. Implications of the Research Study

The development of the heuristic problem-solving treatment, its implementation in a quasi-experimental context (Helmstadter, 1970: 102, 103) and the analysis of the results have revealed both the strengths and the weaknesses of this unit. Based upon the analysis of the results, the opinions of both the students and the cooperating teachers, and the observations of the author, a number of implications will be stated, both for the classroom and for further research.

1. Implications for the Classroom

There are four implications for the classroom that have arisen as a result of this study and would therefore by helpful to teachers who desire to implement the heuristic problem-solving unit as an integral part of their Mathematics 20 program.

The first implication is that the statistical analysis of the results in this study has revealed no significant differences between the heuristic problem-solving treatment and the conventional problem-solving treatment on two tests; one was designed to provide scores on problems directly related to the problems contained in the treatments

(Table 6.14, p. 126), and the other test was designed to provide scores on problems typical of the Mathematics 20 course of studies (Table 6.20, p. 135). Therefore, a regular classroom teacher, who would desire to incorporate a problem-solving chapter into his Mathematics 20 program, could use either of the treatments that the author has developed. It should be noted, however, that the students' reaction to the two treatments, as expressed in the Student Opinion Survey, was extremely positive for the experimental group, and much less so for the control group (pp. 112-116).

Secondly, the author would recommend the insertion of additional lessons in the heuristic problem-solving unit. Tables 6.11 (p. 120) and 6.12 (p. 123) reveal the students' need for additional practice in utilizing the solution method Specialization. Analogy, another method of solution, was also inadequately utilized by the students (Table 6.11, p. 120).

Thirdly, from the author's observations, he would recommend that if the heuristic problem-solving unit were taught in a semestered school, that it be taught in conjunction with the regular curriculum; that is, part of the double period be allotted for the coverage of the regular curriculum and the other part be given to the teaching of this unit.

Fourth, Polya's book, How to Solve It (1957), gives eight methods of solution (generalization, specialization, analogy, decomposing and recombining, working backward, definition, recursion, and auxiliary elements and problems) that can be used to solve appropriate problems. Of these eight methods of solution, the author

chose five (generalization, specialization, analogy, decomposing and recombining, and working backward). Definition was not included because this method was rarely used by itself in solving a problem, but most always was utilized only as a prerequisite to solution in clarifying the problem so that a plan for solving the problem could be devised. Recursion was not included because the application of this method of solution to mathematics problems was beyond the grasp of the average high school student. Finally, auxiliary elements and problems were not included because this method of solution also does not exist independent of the other methods of solution. For example, this method is used in setting up equations which is included within another method of solution, decomposing and recombining. Therefore, a classroom teacher who wants to include all eight solution methods, must develop the additional lessons himself.

The final implication concerns the ease and convenience with which the heuristic problem-solving unit may be utilized in the classroom. The unit contains <u>detailed</u> lesson plans for replication purposes. If a classroom teacher desires to use this unit, the directions provided with each lesson are very explicit. However, this concern for detail makes for much extra reading and this might prevent the classroom teacher from implementing the unit. To overcome this obstacle the author would recommend inservice sessions so that the teacher could be shown the overall strategy for the unit before himself having to delve into the details of each lesson.

2. Implications for Further Research

There are five implications for further research that will provide opportunities to modify and improve the heuristic problem-solving test, to test it on a larger scale, and to improve the generalizability, and therefore the external validity of this experimental study. The first implication pertains to the modification of the experimental treatment and the replication of this thesis in a modified form. The author would like to recommend the insertion of four additional lessons into this unit, one on Specialization, one on Analogy, and two on mixed problems. The summaries of results in Tables 6.11 (p. 120) and 6.12 (p. 123) reveal the students' need for additional practice in utilizing the solution methods, specialization and analogy. A modified format suggested below could be followed:

Lesson 1	Test
Lesson 2-11	Teaching five methods of solution
Lessons 12, 13	Test and discussion
Lessons 14-18	Mixed problems
Lessons 19, 20	Quiz and discussion
Lessons 21-23	Tests

Secondly, the author would recommend that this research study, in its modified form, be repeated using a larger sample; possibly four classes of Mathematics 20 students. A limitation of this study, stated in Chapter I, p. 10, was the lack of generalizability of his sample to the population, that is, all the high schools in Edmonton.

A third implication involves the inservice training of the cooperating teachers. That is, instead of the researcher doing the

teaching himself, the cooperating teachers should be trained to teach the treatments themselves.

Fourth, a major contribution to research in the area of heuristic problem solving would be made if the heuristic problem-solving approach, designed by the researcher, could be implemented at other grade levels. Kilpatrick (1969), in his literature review on problem solving, mentioned the dearth of published studies on problem solving at the secondary school level. For example, many bad habits or mental blocks have already been formed by grade eleven students, so it would be beneficial to attempt implementing this unit at the junior high school level.

Finally, the author would recommend further investigation of the variable 'problem characterization,' in an attempt to discover if successful problem solvers 'cue' on the problem characteristics utilized in this study and if not, to discover what cues are used as problem characteristics.

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APPENDICES

APPENDIX A

PRETEST - POST-TEST: MATHEMATICS 20

PROBLEM-SOLVING TEST

MATHEMATICS 20 PROBLEM-SOLVING TEST

Name: ________________________

Time: 70 minutes

Instructions: There are 5 problems in this test; each is worth equal marks.

Show all your work; marks will be alloted for both the work done (leading to the solution) and for the correct solution.

This test will be administered in the following manner: Each problem will be alloted a specified length of time.

Introduction: 5 minutes

Problem #1 - 10 minutes

Problem #2 - 10 minutes

Problem #3 - 10 minutes

Problem #4 - 15 minutes

Problem #5 - 10 minutes

Review: 10 minutes

The teacher will start and stop the students on each problem.

Problem 1

Three years from now, father will be 11 times as old as his son was 3 years ago. Three years ago, the father was 3 times as old as his son will be 3 years from now. Find the present age of each.

Problem 2

The total value of a certain number of dimes and quarters is \$27.20. If there had been twice as many dimes and half as many quarters, their total value would be \$37.15. How many coins of each kind are there?

Problem 3

The unit's digit of a two-digit number exceeds 3 times the ten's digit by 2. If 13 is added to the original number, the result is one half of the number obtained by reversing the digits. Find the original number.

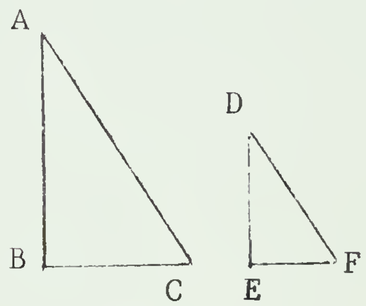
Problem 4

Given a right circular cone with radius 10 cm. and height 30 cm. Find the volume of its frustum given that the height of the frustum is 20 cm.

Hints: #1 $\pi \approx 3.14$

#2 Volume of cone $(1/3)(\pi)r^2h$

#3 Similar triangles

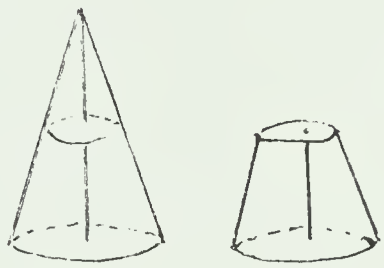


If triangle ABC is similar to triangle DEF, then:

angle A = angle D
angle B = angle E
angle C = angle F

$(\overline{AB}/\overline{DE})=(\overline{BC}/\overline{EF})=(\overline{AC}/\overline{DF})$

#4 The "frustum" is that part of the cone-shaped solid left after the tip has been cut off by a plane parallel to the base.



CONE FRUSTUM

Problem 5

Two cars start from the same point, the first one travels south and the other west. If the first car travels at the rate of 24 miles per hour, what must be the rate of the second car in order that the distance between the two cars at the end of two hours be 52 miles?

APPENDIX B

FIVE-METHOD COMPREHENSION
TEST

Name: ________________

FIVE-METHOD COMPREHENSION TEST

Time: 50 minutes.

Instructions: There are 5 problems in this test; each is worth equal marks.

FOR EACH PROBLEM, THE METHOD OF SOLUTION IS GIVEN. USE IT TO SOLVE THE PROBLEMS.

Show all your work; marks will be alloted for both the work done (leading to the solution) and for the correct solution.

The five problems are similar to the problems in the previously-completed lessons, so they are familiar to the students.

This test will be administered in the following manner Each problem will be alloted a specified length of time.

Problem #1 - 9 minutes

Problem #2 - 9 minutes

Problem #3 - 9 minutes

Problem #4 - 9 minutes

Problem #5 - 9 minutes

Review - 5 minutes

The teacher will start and stop the students on each problem.

Information: 0 is an even number.

Problem 1 (Generalization)

In a triangle ABC, if 25 lines are drawn from the vertex A through points on the opposite side, how many triangles are formed?

Problem 2 (Working Backward)

Given a jar that will hold exactly 7 quarts of water, a jar that will hold 3 quarts of water, no other containers holding water, but an infinite supply of water, describe a sequence of fillings and emptyings of water jars that will result in achieving 5 quarts of water.

Problem 3 (Specialization)

John collects stamps. Currently he has 67 stamps and a stamp book with 12 pages in it. He wants to place an odd number of stamps on the odd-numbered pages, an even number of stamps on the even-numbered pages, and a different number of stamps on each of the 12 pages. Can he do so? How?

Problem 4 (Analogy)

Find the length of the diagonal of a rectangular solid of which the length, width, and height are 12 cm., 4 cm., and 3 cm. respectively.

Problem 5 (Decomposing and Recombining)

The difference between two numbers is 21. If four times the larger is subtracted from 32 times the smaller, the remainder is 28. What are the numbers?

APPENDIX C

HEURISTIC PROBLEM-SOLVING

TEST

Name:

____

HEURISTIC PROBLEM-SOLVING TEST

Time: 75 minutes

Instructions: There are 5 problems in this test; each is worth equal marks.

Show all your work; marks will be alloted for both the work done (leading to the solution) and for the correct solution.

This test will be administered in the following manner: Each problem will be alloted a specified length of time.

Introduction: 5 minutes

Problem #1 - 10 minutes

Problem #2 - 15 minutes

Problem #3 - 15 minutes

Problem #4 - 15 minutes

Problem #5 - 15 minutes

Review: 5 minutes

The teacher will start and stop the students on each problem.

Problem 1

Two airplanes, travelling in opposite directions, started from the same point at the same time. After 4 hours, the two airplanes were 8000 km apart. If one airplane travelled three times as rapidly as the other, what were their rates?

Problem 2

To prove: $1/(2!) + 2/(3!) + 3/(4!) + \ldots + n/((n+1)!) = 1 - n/((2n-1)!)$

Hint: "!" is an exclamation mark.

The meaning of "!" is, for example,
$5! = 5 \times 4 \times 3 \times 2 \times 1$ and
$11! = 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$.

Problem 3

A man goes into a store and says to the proprietor, "Give me as much money as I have with me and I will spend $10.00 in your store". This was done. The man repeated the operation in a second store and a third store, after which he had no money left. How much did he start with?

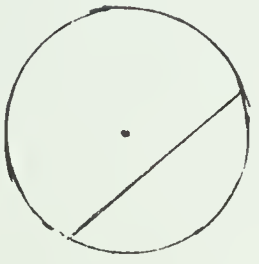
Problem 4

Find the diagonal of a regular octahedron with a given edge 10 cm.

Problem 5

In a circle, find the maximum number of regions formed by 20 chords.

Hints: #1 -



A chord is a straight line extending from one end of an arc of a circle to the other.

#2 - If there is more than one chord in a given circle, they may intersect each other.

APPENDIX D

CHARACTERISTICS AND

SOLUTION-METHODS TEST

CHARACTERISTICS AND SOLUTION-METHODS TEST

Name: ____________________

Time: 20 minutes

Instructions: This test is composed of ten problems.

Its purpose is to examine the students' ability to choose the key characteristics of each problem, and then the correct method of solution to follow in solving each problem.

STUDENTS ARE NOT ASKED TO WORK OUT AND WRITE DOWN A COMPLETE SOLUTION TO EACH PROBLEM but rather to choose a METHOD OF SOLUTION that they feel will lead to the correct solution to the problem.

Below, are listed the five possible methods of solution (with accompanying definitions).

A. Generalization

This method of solution involves a consideration of specific instances to a consideration of an inference, that what holds true for the specific will also hold true for 'n'.

B. Specialization

This method of solution entails the consideration of some specialized case which either, in itself solves the problem, or provides the strategy (that used to solve the specialized case) to solve the problem.

C. Analogy

This method of solution involves a consideration of two systems which agree in clearly definable relations between their respective parts. (For instance, a triangle in a plane is analogous to a tetrahedron in space).

D. Decomposing and Recombining

This method of solution entails decomposing a problem into its component parts and then recombining the parts into a more or less different whole.

E. Working Backward

This method of solution involves starting from what is required and then assuming what is sought as already found.

Problem 1

Two airplanes, travelling in opposite directions, started from the same point at the same time. After 4 hours, the two airplanes were 8000 km. apart. If one airplane travelled three times as rapidly as the other, what were their rates?

(a) Write down the characteristics of this problem.

(b) Underline the correct method of solution for this problem.
Generalization Specialization Analogy Decomposing and Recombining Working Backward

Problem 2

To prove: $1/(2!) + 2/(3!) + 3/(4!) + \ldots + n/((n+1)!) \quad 1 - n/((2n-1)!)$

Hint: "!" is an exclamation mark.

The meaning of "!" is, for example,
$5! = 5x4x3x2x1$ and
$11! = 11x10x9x8x7x6x5x4x3x2x1$.

(a) Write down the characteristics of this problem.

(b) Underline the correct method of solution for this problem.
Generalization Specialization Analogy Working Backward
Decomposing and Recombining

Problem 3

A man goes into a store and says to the proprietor, "Give me as much money as I have with me and I will spend $10.00 in your store". This was done. The man repeated the operation in a second store and a third store, after which he had no money left. How much did he start with?

(a) Write down the characteristics of this problem.

(b) Underline the correct method of solution for this problem.
Generalization Specialization Analogy Working Backward
Decomposing and Recombining

Problem 4

Find the diagonal of a regular octahedron with a given edge 10 cm.

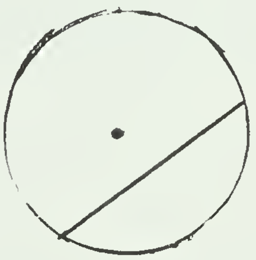
(a) Write down the characteristics of this problem.

(b) Underline the correct method of solution for this problem.
Generalization Specialization Analogy Working Backward
Decomposing and Recombining

Problem 5

In a circle, find the maximum number of regions formed by 20 chords.

Hints: #1 -



A chord is a straight line extending from one end of an arc of a circle to the other.

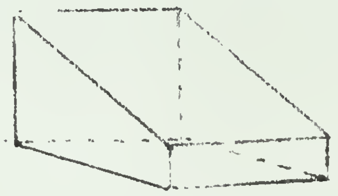
#2 - If there is more than one chord in a given circle, they may intersect each other.

(a) Write down the characteristics of this problem.

(b) Underline the correct method of solution for this problem.
Generalization Specialization Analogy Working Backward
Decomposing and Recombining

Problem 6

The janitor of a local school was asked if he had ever encountered problems in his job that required mathematical solutions. His reply was that there were several, but one in particular that he could not solve. That one was: If there was a door stop, as shown below, what would be the length of the line joining the two corners which are furthest apart?



(a) Write down the characteristics of the problem.

(b) Underline the correct method of solution for this problem.
Generalization Specialization Analogy Working Backward
Decomposing and Recombining

Problem 7

The length of each of the congruent sides of an isosceles triangle is $1\frac{1}{2}$ times the length of the base. The perimeter of the triangle is 60 cm. Find the length of each side of the triangle.

(a) Write down the characteristics of the problem.

(b) Underline the correct method of solution for this problem.
Generalization Specialization Analogy Working Backward
Decomposing and Recombining

Problem 8

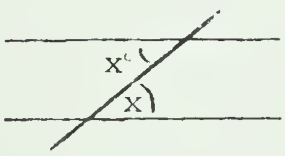
The tower of Hanoi problem consists of 8 discs of graduated size and moving them among three pegs. The problem is to transfer the 8 discs from one spot to another in the fewest possible moves, moving one disc at a time and never placing a disc on top of a smaller one. How many moves will it take?

(a) Write down the characteristics of the problem.

(b) Underline the correct method of solution for this problem.
Generalization Specialization Analogy Working Backward
Decomposing and Recombining

Problem 9

You are given the following: (a) A straight line equals an angle of 180°; (b) A right angle equals 90°; (c) If two parallel lines are cut by a transversal, the alternate interior angles are equal.



Prove that the sum of the angles of any triangle equals 180°.

(a) Write down the characteristics of the problem.

(b) Underline the correct method of solution for this problem.
Generalization Specialization Analogy Working Backward
Decomposing and Recombining

Problem 10

Lay 21 toothpicks in a row. The object of the game is to make your opponent pick up the last toothpick by following these rules.

1. Players take alternate turns.
2. Each turn a player picks up 1, 2, or 3 toothpicks as he chooses.
3. The player who must pick up the last toothpick loses.

Is this just a game of chance, or can it be played scientifically?

(a) Write down the characteristics of the problem.

(b) Underline the correct method of solution for this problem.
Generalization Specialization Analogy Working Backward
Decomposing and Recombining

APPENDIX E

STUDENT OPINION SURVEY

STUDENT OPINION SURVEY

1. Did you find that the procedure presented (eg. Unknown, Given, etc.) was helpful in solving problems? Why or why not?

   ____

   ____

   ____

2. What method of solution did you feel most comfortable with?

   ____

3. What method of solution did you feel least comfortable with?

   ____

4. Do you feel that you are a better problem solver now than before? Why or why not?

   ____

   ____

   ____

5. Do you think this unit will help you in solving new problems in Mathematics 20? If yes, how? If no, why not?

   ____

   ____

   ____

6. Other comments and concerns

   ____

   ____

   ____

APPENDIX F

TEACHER OPINION SURVEY

TEACHER OPINION SURVEY

1. As an experienced teacher yourself, do you feel that other Mathematics 20 teachers would want to make use of this method of problem solving?

____

____

2. Do you feel that the students benefited by learning to use this method of problem solving?

____

____

3. What did you like, in particular, about this method of problem solving?

____

____

____

4. What did you dislike (if anything) about this method of problem solving?

____

____

5. Please make other comments that you wish to.

____

____

APPENDIX G

LESSON PLANS FOR THE HEURISTIC APPROACH

TO PROBLEM SOLVING

1. Lesson One

(a) Lesson Plan

Time	Topic
	Mathematics 20 Problem-Solving Test (pretest)

10 min. Introduction

The teacher will introduce the unit, telling the students what is its primary purpose. He will tell them briefly about the format of the project; that is, that this unit will span nineteen class periods. The instructor will mention that the students will be given tests on the first, eleventh, seventeenth, eighteenth, and nineteenth days, and that they will be taught fourteen lessons on problem solving inbetween the first and seventeenth days.

(The teacher must be certain that the students have every problem and every solution written in their notebooks for lessons two through sixteen. Students who are absent will be told to borrow someone else's notebook and to copy the problems and solutions that they missed, into their own notebooks).

The instructor will tell the students that his approach to problem solving is through demonstration followed

by imitation. That is, the teacher will demonstrate the first problem involving a particular method of solution and then the students will attempt to solve subsequent problems themselves, by imitation.

70 min. <u>Administration of the Test</u>

The instructor will hand out the tests and in doing so, he will tell the students to fill in their names on the first page of the test. (The test is contained in this thesis in Appendix A).

(b) Scoring Procedure For Mathematics 20 Problem-Solving Test

Problem 1 (6 marks)

Let x years be father's present age.

Let y years be son's present age.

(1 mark for naming both unknowns. If the units are missing after both x and y, then minus $\frac{1}{2}$ mark).

$x+3 = 11(y-3)$ (Call this equation #1)

$x-3 = 3(y+3)$ (Call this equation #2)

(2 marks; 1 mark for each equation)

$x-11y = -36$ (Expansion of equation #1)

$x-3y = 12$ (Expansion of equation #2)

(1 mark for the above system of expanded equations)

$8y = 48$

$y = 6$ years (1 mark; minus $\frac{1}{2}$ mark if units are missing)

Solve for x in equation #1

$x-11(6) = -36$

$x = 30$ years (1 mark; minus $\frac{1}{2}$ mark if units are missing).

Problem 2 (6 marks)

Let x be the number of dimes

Let y be the number of quarters

(1 mark; if "number" or "value" are not specified in the above statements, then minus $\frac{1}{2}$ mark)

$10x + 25y = 2720$ (1 mark)

$20x + (25/2)y = 3715$ (1 mark)

Multiply the first equation by 2

$$20x + 50y = 5440$$

(1 mark for establishing a reducible system of equations)

$$(75/2)y = 1725$$

$$y = 46 \text{ quarters} \quad \text{(1 mark)}$$

Solve for x in the first equation

$$10x + 25(46) = 2720$$

$$10x = 1570$$

$$x = 157 \text{ dimes} \quad \text{(1 mark)}$$

Check the solutions in the second equation

$$20(157)+(25/2)(46) = 3715$$

$$3715 = 3715$$

Therefore, there are 157 dimes and 46 quarters.

(In the above two solutions, the appropriate units must accompany the numerical values, otherwise minus ½ mark for each missing unit)

Problem 3 (6 marks)

Let x be the 10's digit.

Let y be the 1's digit.

(1 mark for naming the two unknowns)

$$y = 3x+2 \quad \text{(Call this equation \#1)}$$

$$10x+y+13 = \tfrac{1}{2}(10y+x) \quad \text{(Call this equation \#2)}$$

(2 marks; 1 mark for each equation)

$$y = 3x+2 \quad \text{(Equation \#1)}$$

$$19x-8y = -26 \quad \text{(Simplification of equation \#2)}$$

(1 mark if student has the above simplified system of equations)

$$19x-8(3x+2) = -26 \quad \text{(Substitution of equation \#1 into \#2)}$$

$$x = 2 \quad \text{(1 mark)}$$

Solve for y in equation #1.

$$y = 3(2)+2$$

$$y = 8 \quad \text{(1 mark)}$$

Check the solutions in equation #2.

$$19(2)-8(8) = -26$$

$$-26 = -26$$

Therefore the two-digit number is 28.

<u>Problem 4</u> (6 marks)

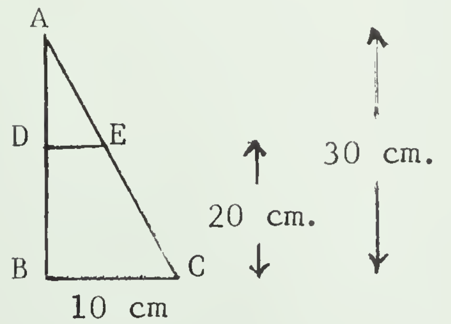
Volume of larger cone:

$$\text{Volume} = (\pi)r^2h/3$$

$$= (3.14)(10)^2(30)/3 \quad (\tfrac{1}{2}\text{ mark})$$

$$= 3140 \text{ cm.}^3 \quad (\tfrac{1}{2}\text{ mark})$$

Radius of smaller cone:



Triangle ABC is similar to triangle ADE

$(\overline{AD}/\overline{AB}) = (\overline{DE}/\overline{BC})$

$(10/30) = (r/10)$

Therefore $r = 3.33$ cm.

(1 mark for correct r)

(1 mark for correct ratios)

Volume of smaller cone:

$$\text{Volume} = (\pi)r^2h/3$$

$$= (3.14)(3.33)^2(10)/3$$

$$= 116.1 \text{ cm.}^3 \quad \text{(1 mark)}$$

Volume of the frustum:

$$\text{Volume} = \text{(Volume of larger cone) - (Volume of smaller cone)}$$

$$= (3140) - (116.1)$$

$$= 3023.9 \text{ cm.}^3$$

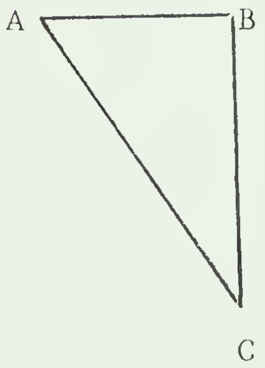
(1 mark for correct formula for frustum)

(1 mark for correct answer; minus ½ mark if units are missing)

Problem 5 (6 marks)

Let x mph be the rate of the second car (1 mark)

d = 48 miles



r = x mph
d = 2x miles

$$(\overline{AC})^2 = (\overline{AB})^2 + (\overline{BC})^2$$

$$52^2 = 48^2 + (2x)^2$$

(2 mark for the correct equations; 1 mark each)

($\frac{1}{2}$ mark for 48 miles)

($\frac{1}{2}$ mark for 2x miles)

$$x = \sqrt{(52^2-48^2)}/2$$

$$= 10 \text{ mph}$$

(2 marks; minus $\frac{1}{2}$ mark if units are missing; if left as 20 mph instead of 10 mph, only 1 mark)

(If x was used in the equation above instead of 2x, minus $\frac{1}{2}$ mark)

2. Lesson Two

Time	Topic
	Problem Solving (Generalization)
25 min.	Instructional Strategy For Problem One

The instructional strategy that would be used in solving this problem is as follows. The teacher will write the problem on the blackboard and then pause for several minutes allowing the students sufficient time to understand the problem. He will then begin by asking: What is the unknown? He will answer that the unknown is the resultant number of triangles that are formed. The second question will be: What information is given? The given data are: a triangle ABC, 20 lines drawn from vertex A through points on the opposite side. Since the unknown and the data have been delineated, it remains to devise some plan by which the instructor might proceed from the given data to the unknown.

The teacher will then ask the following question: Do you know a related problem? The purpose of this question is to get the problem solver to discover a method of solution that has a reasonable chance of leading to the correct solution. Before responding to this

question, the teacher will name the characteristics of the problem currently before him. He will then think of a related problem (with similar characteristics) and name its method of solution.

Since this problem is characterized as a problem involving pattern development, the method of solution decided upon by the instructor is generalization. The teacher will then ask himself the question: Do you know a related problem? A simple related problem that he will give is:

> If 0 lines are drawn from vertex A through 0 points of the opposite side, how many triangles are formed?

He will then ask if there is another related problem that would help in solving the original problem. The answer that the instructor will give is to present the second related problem:

> If 1 line is drawn from vertex A through 1 point of the opposite side, how many triangles are formed?

He will then ask if there is another related problem that would help in solving the original problem. The answer that the teacher will give is to present the third related problem:

> If 2 lines are drawn from vertex A through 2 points of the opposite side, how many triangles are formed?

He will then ask if there is another related problem that

would help in solving the original problem. The answer that the instructor will give is to present the fourth related problem:

> If 3 lines are drawn from vertex A through 3 points of the opposite side, how many triangles are formed?

Next, the teacher will ask if these four related problems could help them in solving the original problem. He will draw the table as shown below and ask if the students could see a pattern developing that would lead to the solution of the problem.

Number of Lines	Number of Triangles
0	1
1	3
2	6
3	10
.	.
.	.
.	.
20	?

It should be noted that pictures of triangles corresponding to each of the four related problems will be drawn on the blackboard as each related problem is presented. This will aid in clarifying the four related problems used in devising the plan. The teacher will then expand the table to include all the possibilities,

0-20, for the number of lines drawn. (The steps used in devising a plan and the table used in carrying out the plan are all presented in this thesis under the heading: "Answers to Problems From the Second Day").

After the plan has been carried out and a solution has been reached, the teacher will ask the questions:

> Is the result correct?
>
> What are the characteristics of this problem that could be associated with the method of solution that was utilized?

The instructor will first check his result to ensure that it is correct. After he has done that, he will examine the problem again in order to note its characteristics. He will then write them on the blackboard (along with the method of solution used to correctly solve this problem). The characteristics are:

1. The number of lines is large; the estimated answer is very large.

2. To draw a diagram depicting the situation, and to attempt to count all the resulting triangles is an extremely difficult task.

15 min. Instructional Strategy For Problem Two

The teacher, after demonstrating how to solve problem one, will then ask the students to try problem two by

themselves. After about ten minutes the instructor will take up the problem with them. He will ask the same series of questions that he asked for problem one, that is:

> What is the unknown?
> What information is given?
> Do you know a related problem?

In this second problem, the teacher will attempt to have the students give the answers to these questions. If the students have difficulty in answering the first two questions, the instructor will assist them. Before they respond to the third question, the teacher will ask them to name the characteristics of the problem. He will then ask them if they were able to think of a related problem, and if so, to name its method of solution. (Since this problem is characterized as a problem involving pattern development, the method of solution decided upon by the students and/or the teacher is generalization).

The instructor will then ask the students: Do you know a related problem? He will then help them by presenting a related problem (if the students did not respond with the answer to this question themselves). This process will continue until four related problems are presented.

Next, the teacher will ask if these four related

problems could help them in solving the original problem. He will then draw the table as shown below and ask if the students could see a pattern developing that would lead to the solution of the problem.

Number of Points	Number of Lines
2	1
3	3
4	6
5	10
.	.
.	.
.	.
14	?

It should be noted that diagrams corresponding to each of the four related problems will be drawn on the blackboard as each related problem is presented. This will aid in clarifying the four related problems used in devising the plan. The instructor will then expand the table to include all the possibilities, 2-14, for the number of points drawn. (The related problems used in devising a plan and the table used in carrying out the plan are all presented in this thesis under the heading: "Answers to Problems From the Second Day").

After the plan has been carried out and a solution has been reached, the teacher will ask the questions:

> Is the result correct?
>
> What are the characteristics of this problem that could be associated with the method of solution that was utilized?

The instructor will first check the result to ensure that it is correct. After he has done that, he will ask the students again, what are the characteristics of the problem. He will then write them on the blackboard (along with the method of solution used to correctly solve this problem). The characteristics are:

1. The number of points is large; the estimated answer is very large.

2. To draw a diagram depicting the situation, and to attempt to count all the resulting lines is an extremely difficult task.

SECOND DAY

Problem 1

In a triangle ABC, if 20 lines are drawn from vertex A through points on the opposite side, how many triangles are formed?

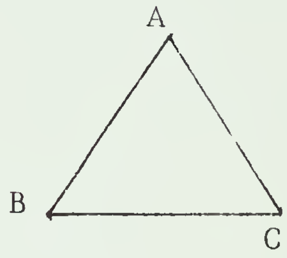
Problem 2

Find the number of lines determined by 14 points, no 3 of which are collinear.

ANSWERS TO PROBLEMS FROM THE SECOND DAY

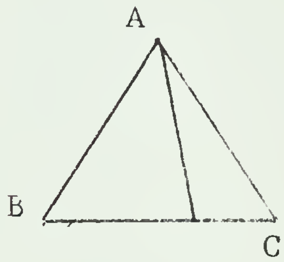
Problem 1

Case #1 -



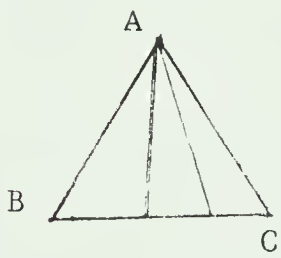
If 0 lines are drawn from vertex A through 0 points of the opposite side, there is 1 triangle that is formed.

Case #2 -



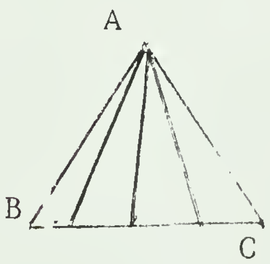
If 1 line is drawn from vertex A through 1 point of the opposite side, there are 3 triangles that are formed.

Case #3 -



If 2 lines are drawn from vertex A through 2 points of the opposite side, there are 6 triangles that are formed.

Case #4 -



If 3 lines are drawn from vertex A through 3 points of the opposite side, there are 10 triangles that are formed.

Number of Lines	Number of Triangles
0	1
1	3
2	6
3	10
4	15
5	21
6	28
7	36
8	45
9	55
10	66
11	78
12	91
13	105
14	120
15	136
16	153
17	171
18	190
19	210
20	231

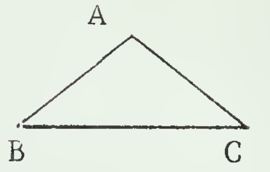
Therefore, if 20 lines are drawn from the vertex A through points on the opposite side, there are 231 triangles that are formed.

However, ideally the problem solver would only have to calculate the triangles formed for 0, 1, 2, and 3 lines. From there, through a process of judicious guessing (that is, through what Polya called "trial and error" and the "bright idea"), the problem solver could derive the formula $n(n+3)/2 + 1$, that expresses the number of triangles formed, given n lines.

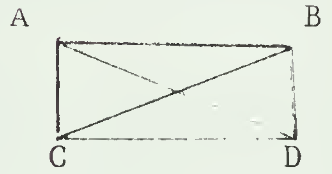
Problem 2

Case #1 - 

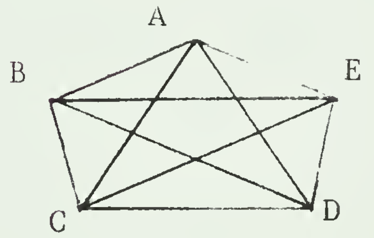
Given 2 points, 1 line can be drawn.

Case #2 - 

Given 3 points, 3 lines can be drawn.

Case #3 - 

Given 4 points, 6 lines can be drawn.

Case #4 - 

Given 5 points, 10 lines can be drawn.

Number of Points	Number of Lines
2	1
3	3
4	6
5	10
6	15
7	21
8	28
9	36
10	45
11	55
12	66
13	78
14	91

Therefore, given 14 points, there are 91 lines that can be drawn.

However, ideally the problem solver would only have to calculate the lines formed for 2,3,4, and 5 points. From there, through a process of judicious guessing (that is, through what Polya called "trial and error" and the "bright idea"), the problem solver could derive the formula $n(n-1)/2$, that expresses the number of lines formed, given n points.

3. Lesson Three

Time	Topic
	Problem Solving (Generalization)
5 min.	Introduction The teacher will begin the class by reviewing, briefly, the strategy for solving problems as it was introduced in lesson two.
25 min.	Instructional Strategy For Problems One and Two Next, the teacher will write the first two problems from today's lesson on the blackboard and will ask the students to try solve both these problems by themselves. After about fifteen minutes, the instructor will ask two students to come up to the blackboard and write out their solution, one pupil to each problem. When the two solutions have been written on the blackboard, the two students will be asked to take their seats. The teacher will then ask the students if there are any corrections that need to be made with regard to the first two questions in each solution. (The first two questions being: What is the unknown? What is the given information?). If there are any

errors and if the pupils have difficulty in locating them, then the instructor will assist them by making the corrections.

The teacher will then examine their responses to the question: Do you know a related problem? Before responding to this question, the instructor will want the students to name the characteristics of the two problems currently before them. The teacher will then examine the suggested methods of solution given on the blackboard. If both solution methods are correct, he will then check to see that the two students had characterized the problems correctly. If however, either one or both solutions on the blackboard, used an incorrect method of solution, then the instructor will complete the solution(s) for the students.

For Problem One:

Since this problem was characterized by the students as a problem involving pattern development, the method of solution decided upon by the students and/or the teacher, is generalization. (The related problems used in devising a plan and the table used in carrying out the plan, are all presented in this thesis under the heading: "Answers to Problems From the Third Day").

Finally, the instructor will examine the solution to

problem one in order to ascertain how the student answered the question: What are the characteristics of this problem that could be associated with the method of solution that was utilized? The characteristics are:

1. The number of rays is large; the estimated answer is very large.

2. To draw a diagram depicting the situation, and to attempt to count all the resulting angles is an extremely difficult task.

For Problem Two:

Since this problem was characterized as a problem involving pattern development, the method of solution decided upon by the students and/or the teacher, was generalization. (The related problems used in devising a plan and the table used in carrying out the plan, are all presented in this thesis under the heading: "Answers to Problems From the Third Day").

Finally, the instructor will examine the solution to problem two in order to ascertain how the student answered the question: What are the characteristics of this problem that could be associated with the method of solution that was utilized? The characteristics are:

1. The number of odd numbers is large; the estimated answer is very large.

2. To add all the odd numbers together is an extremely difficult and laborious task.

10 min.

Instructional Strategy For Problem Three

The teacher will write the third question on the blackboard and allow the students about only about five minutes to work on this problem. He will then correct it in class. (The related problems used in devising a plan and the table used in carrying out the plan, are all presented in this thesis under the heading: "Answers to Problems From the Third Day").

THIRD DAY

Problem 1

How many angles are formed when 15 rays are drawn from the same end point?

Problem 2

Find a way to determine the sum of the odd numbers less than 100.

Problem 3

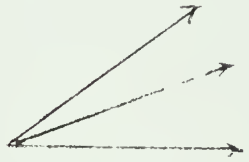
```
            E
          E S
        E S K
      E S K I
    E S K I M
  E S K I M O
E S K I M O S
```

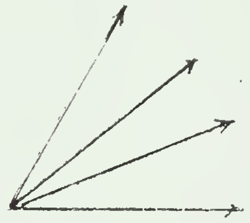
Given the grid, as shown on the left, how many different ways can the word ESKIMOS be made? (The word is made by moving in a vertical and/or horizontal direction on the grid).

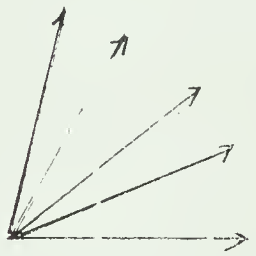
ANSWERS TO PROBLEMS FROM THE THIRD DAY

Problem 1

Case #1 -  Given 2 rays, there are 2 angles formed.

Case #2 -  Given 3 rays, there are 6 angles formed.

Case #3 -  Given 4 rays, there are 12 angles formed.

Case #4 -  Given 5 rays, there are 20 angles formed.

Number of Rays	Number of Angles Formed
2	2
3	6
4	12
5	20
6	30
7	42
8	56
9	72
10	90
11	110
12	132
13	156
14	182
15	210

Therefore, there are 210 angles formed.

However, ideally the problem solver would only have to calculate the number of angles formed using 2,3,4, and 5 rays. From there, through a process of judicious guessing (that is, through what Polya called "trial and error" and the "bright idea"), the problem solver could derive the formula $n(n-1)$, that expresses the number of angles formed, given n rays.

Problem 2

Case #1 - $1 = 1$ — Given the first odd number, the sum is the first perfect square.

Case #2 - $1+3 = 4 = 2^2$ — Given the first two odd numbers, the sum is the second perfect square.

Case #3 - $1+3+5 = 9 = 3^2$ — Given the first three odd numbers, the sum is the third perfect square.

Case #4 - $1+3+5+7 = 16 = 4^2$ — Given the first four odd numbers, the sum is the fourth perfect square.

Number of Odd Numbers Added	Sum of the Odd Numbers
1	1
2	$4=2^2$
3	9
4	$16=4^2$
.	
.	
.	
50	50^2

Therefore, given the first 50 odd numbers, their sum is 50^2 or 2500.

However, ideally the problem solver would only have to calculate the resulting sum using the first 1,2,3, and 4 odd numbers. From there, through a process of judicious guessing (that is, through what Polya called "trial and error" and the "bright idea"), the problem solver could derive the formula n^2, that expresses the sum, given the first n odd numbers.

Problem 3

Case #1 - E — Given 1 letter on the grid, there is 1 word that can be made.

Case #2 -
```
  E
E S
```
Given 2 letters on the grid, there are 2 words that can be made.

Case #3 -
```
    E
  E S
E S K
```
Given 3 letters on the grid, there are 4 words that can be made.

Case #4 -
```
      E
    E S
  E S K
E S K I
```
Given 4 letters on the grid, there are 8 words that can be made.

Number of Letters Used	Number of Words Made
1	1
2	2
3	4
4	8
5	16
6	32
7	64

Therefore, given 7 letters on the grid, there are 64 words that can be made.

However, ideally the problem solver would only have to calculate the number of words formed using 1,2, 3, and 4 letters on the grid. From there, through a process of judicious guessing, the problem solver could derive the formula 2^{n-1}, that expresses the number of words formed given n letters on the grid.

4. Lesson Four

Time	Topic
	Problem Solving (Working Backward)

25 min.

Instructional Strategy For Problem One

The teacher will write the problem on the blackboard and then pause for several minutes allowing the students sufficient time to understand the problem. The instructor will begin by asking: What is the unknown? He will answer that the unknown is to fill the nine-quart pail with six quarts of water. The second question will be: What information is given? The given data are: an unlimited supply of water, a four-quart pail (unmarked with any quantity scale), and a nine-quart pail (also unmarked).

Since the unknown and the data have been delineated, it remains to devise some plan by which the instructor might proceed from the given data to the unknown. The teacher will then ask the following question: Do you know a related problem? The purpose of this question is to get the problem solver to discover a method of solution that has a reasonable chance of leading to the correct solution. Before responding to that question, the teacher will name the characteristics of the problem currently

before him. He will then think of a related problem (with similar characteristics) and name its method of solution. Since this problem is characterized as a problem in which there are a large number of paths that lead away from the initial state but only one that reaches the given goal state, the method of solution decided upon by the teacher, is working backward.

The instructor will then ask the question: Can you restate the problem? The answer that the teacher will give is to restate the problem:

> Given that you have a four-quart pail and a nine-quart pail, and also given that your nine-quart pail already has six quarts of water in it, how did you get it there?

The instructor will then start at the end of the solution and work backward. Questions similar to the following (with corresponding replies) will be used to devise a plan or strategy.

1. Question: "If you have six quarts of water in the nine-quart pail, where did it come from?"

   Reply: "The water came from the nine quarts of water in the full pail of which three quarts were removed".

2. Question: "How were the three quarts removed?"

   Reply: "The water (three quarts) was poured out of the full nine-quart pail into the four-quart pail in which there was already one quart".

3. Question: "Where did the one quart of water, in the four-quart pail, come from?"

Reply: "The one quart of water came as a result of pouring out eight quarts of water (filling the four-quart pail twice) from a full nine-quart pail. Then pouring the remaining one quart into the empty four-quart pail".

Now that a plan has been devised, carry it out, starting at the beginning of the problem. Take a nine-quart pail, full of water, and pour out eight quarts (filling the four-quart pail twice). Then pour the remaining one quart into the empty four-quart pail. Next, refill the nine-quart pail. Pour out the water from the nine-quart pail so as to fill the four-quart pail. There are now only six quarts remaining in the nine-quart pail. It should be noted that pictures of pails, with their corresponding quantities of water, will be drawn on the blackboard at appropriate times throughout the problem. This will aid in clarifying the questions and their replies used in devising a plan. (The diagrams that will be used in carrying out the plan are presented in this thesis urder the heading: "Answers to Problems From the Fourth Day").

After the plan has been carried out and a solution has been reached, the instructor will ask the questions:

Is the result correct?

What are the characteristics of this problem that could be associated with the method of solution that was utilized?

The teacher will first check his result to ensure that it is correct. After he has done that, he will examine the problem again in order to note its characteristics. He will write them on the blackboard (along with the method of solution used to correctly solve this problem). The characteristics are:

1. The outcome is known (six quarts of water in the nine-quart jar).

2. The problem has a trial-and-error beginning.

15 min.

Instructional Strategy For Problem Two

The teacher, after demonstrating how to solve problem one, will then ask the students to try problem two; to imitate the method that the instructor used to solve the previous problem. If there is enough time, the teacher will take up the problem with them; if not, they will discuss it in the next class. He will ask the same two questions that he asked in problem one: What is the unknown? What information is given? In this problem, the teacher will attempt to have the students give the

answers to these questions. He will write the answers on the blackboard as the pupils call them out. If the students did not respond with the answers to these questions, then he will help them.

Next, the teacher will ask the question: Do you know a related problem? Before attempting to answer this question, the instructor will ask the students for the characteristics of this problem. Based on these characteristics, the teacher will solicit answers to his question. The answer that should be given is problem one, and thus the method of solution that should have been used to solve problem two is working backward.

Since the method of solution to be used in solving the present problem is working backward, the instructor will seek an answer to the question: Can you restate the problem (without changing its meaning)? The restated problem is:

> Fifteen pennies are placed on a table in front of two players. Each player is allowed to remove at least one penny but not more than five pennies at his turn. The players alternate turns, each removing from one to five pennies n number of turns, until one player takes the last penny on the table, and wins all 15 pennies. Given the player who will win the game, is faced with between one and five pennies on the table, and it is his turn. How will he consistently arrive at this position?

The teacher will then solicit answers from the students, as to how they started at the end of the problem and

worked backwards. He will provide them with any steps that they cannot give to him. (The steps and diagrams that were used in devising a plan and in carrying it out are all presented in this thesis under the heading: "Answers to Problems From the Fourth Day").

After the plan has been carried out and a solution has been reached, the teacher will ask the questions:

> Is the result correct?
>
> What are the characteristics of this problem that could be associated with the method of solution that was utilized?

The instructor will first check the result to ensure that it is correct. After he has done that, he will ask the students again, what are the characteristics of the problem. He will then write them on the blackboard (along with the method of solution used to correctly solve this problem). The characteristics are:

1. The outcome is known (the winning player faces one to five pennies on the table, and it is his turn).

2. The problem has a trial-and-error beginning.

FOURTH DAY

Problem 1

Given a jar that will hold exactly 9 quarts of water, a jar that will hold exactly 4 quarts of water, no other containers holding water, but an infinite supply of water, describe a sequence of fillings and emptyings of water jars that will result in achieving 6 quarts of water.

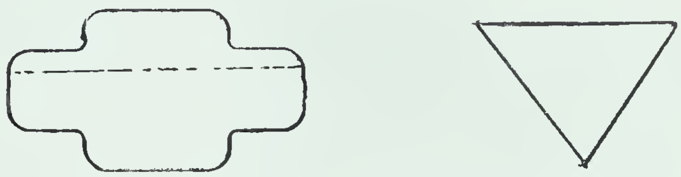
Problem 2

Fifteen pennies are placed on a table in front of two players. Each player is allowed to remove at least one penny but not more than five pennies at his turn. The players alternate turns, each removing from one to five pennies n number of turns, until one player takes the last penny on the table, and wins all 15 pennies. Is there a method of play that will guarantee victory? If so, what is it?

ANSWERS TO PROBLEMS FROM THE FOURTH DAY

Problem 1

Below are the three diagrams that accompany the dialogue of questions and their replies contained in the lesson.



9-quart jar 4-quart jar

First Diagram

9-quart jar 4-quart jar

Second Diagram

9-quart jar 4-quart jar

Third Diagram

Problem 2

Questions similar to the following (with corresponding replies) will be used in order to devise a plan or strategy by which the problem may be solved.

1. Question: "If the ultimate winner wants to face one to five pennies, how many pennies should he leave the ultimate loser confronted with?"

   Reply: "If the ultimate loser was confronted with six pennies, then no matter how many pennies he took (from one to five), there would still be from one to five pennies left on the table, giving the ultimate winner, his turn and thus the victory".

2. Question: "If the ultimate winner wants to face seven to eleven pennies, how many pennies should he leave the opposing player confronted with?"

   Reply: "If the ultimate loser was confronted with twelve pennies, after the ultimate winner's preceding move, then no matter how many pennies the opposing player took (from one to five), the ultimate winner would be able to take enough pennies to confront him with six pennies on his next turn".

3. Question: "What is the ultimate winner's first move?"

   Reply: "He will remove three pennies, leaving the ultimate loser confronted with twelve pennies".

TURN	ULTIMATE WINNER'S TURN	ULTIMATE LOSER'S TURN	NUMBER OF PENNIES CONFRONT-ING HIM	NUMBER OF PENNIES THAT HE WILL CHOOSE
1st	X		15	3
2nd		X	12	1-5
3rd	X		7-11	1-5
4th		X	6	1-5
5th	X		1-5	1-5
6th		X	0	1-5

Therefore, in order to guarantee victory, the above format must be followed.

5. Lesson Five

Time	Topic
	Problem Solving (Working Backward)

10 min.

Problem Two From the Fourth Day

The teacher will discuss problem two from the previous day (if this was not done in class the previous day). The instructional strategy that will be used to discuss this problem is given in Lesson Four under the heading: "Instructional Strategy For Problem Two".

30 min.

Instructional Strategy For Problem One

The instructor will let the students work on problem one. He will ask them in solving this problem to imitate the way in which he had solved problem one (from Lesson Four). The same format will be used to discuss the solution to this problem as was used to discuss problem two (from Lesson Four).

The characteristics of this problem are:

1. The outcome is known (each of the three players has $8).

2. The problem has a trial-and-error beginning.

FIFTH DAY

Problem 1

Three people play a game in which one person loses and two people win each game. The one who loses must double the amount of money that each of the other two players has at that time. The three players agree to play three games. At the end of the three games, each player has lost one game and each person has $8. What was the original stake of each player?

ANSWER TO THE PROBLEM FROM THE FIFTH DAY

Do you know a related problem?

The students should note the characteristics of this problem. Based on these characteristics, they should recall either problem 1 or problem 2 (as the related problem), which in turn should reveal working backward as the method of solution to try on this problem.

Can you restate the problem (without changing its meaning)?

The end of the problem is already solved, therefore there is no need to restate the problem.

Label the first losing player P_1, the second P_2, and the third P_3.

At the end of game 3, P_1, P_2, and P_3 each had \$8. Working backward to the end of game 2, P_1 must have had \$4 and P_2 \$4, since both won in game 3 (P_3 lost), and thus both had their stakes doubled by the results of game 3. Since P_1 and P_2 each gained \$4 in game 3, P_3 must have lost \$8 in game 3, so P_3 had \$16 at the end of game 2.

Working backward to the end of game 1, P_1 must have had \$2 and P_3 \$8, since both won in game 2 (P_2 lost), and thus both had their stakes doubled by the results of game 2. P_2 must have lost \$10 (\$2+\$8), and therefore had \$14 at the end of game 1.

Working backward to the beginning of game 1, P_2 must have had \$7, and P_3 \$4, since both won in game 1 (P_1 lost), and thus both had their stakes doubled by the

results of game 1. P_1 must have lost \$11 (\$4+\$7), and therefore had \$13 at the beginning of game 1. The complete solution obtained by working backward is shown in the following table.

GAME	P(1)	P(2)	P(3)
END OF GAME #3	\$8	\$8	\$8
END OF GAME #2	\$4	\$4	\$16
END OF GAME #1	\$2	\$14	\$8
BEGINNING OF GAME	\$13	\$7	\$4

6. Lesson Six

Time	Topic
	Problem Solving (Specialization)
15 min.	Instructional Strategy For Problem One

The teacher will write the problem on the blackboard and then pause for several minutes allowing the students sufficient time to understand the problem. The instructor will then begin by asking: What is the unknown? He will answer that the unknown is the distribution of the 44 coins into 10 pockets (each pocket must contain a different number of coins). The second question that he will ask is: What information is given? He will answer that the given data are 44 coins, 10 pockets, and the distribution of the coins (a different number of coins are to be placed in each pocket).

Since the unknown and the data have beeen delineated, it remains to devise some plan by which the teacher might proceed from the given data to the unknown. The instructor will then ask the following question: Do you know a related problem? The purpose of this question is to get the problem solver to discover a method of solution that has a reasonable chance of leading to the correct solution. Before responding to that question,

the teacher will name the characteristics of the problem currently before him. He will then think of a related problem (with similar characteristics) and name its method of solution. Since this problem is characterized as a problem that focuses on a particular case, that of finding the minimum number of silver dollars necessary to satisfy the requirement of the problem, the method of solution decided upon by the teacher is specialization.

The instructor will then ask the question: Do you know a more specialized problem? The teacher will then write the special problem on the blackboard.

> What is the minimum number of silver dollars that could be placed into 10 pockets?

He will then solve the more specialized problem and thus demonstrate that it is impossible for Bob to place a different number of coins into each of 10 pockets, given that he only has 44 coins.

First pocket gets 0 coins

Second pocket gets 1 coin

Third pocket gets 2 coins

.
.
.

Tenth pocket gets 9 coins

The minimum number of coins = 1+2+3+ ... +9 = 45 coins.

Therefore, since Bob only has 44 coins, he cannot put a

different number into each pocket.

After the plan has been carried out and a solution has been reached, the teacher will ask the questions:

> Is the result correct?
>
> What are the characteristics of this problem that could be associated with the method of solution that was utilized?

The instructor will first check his result to ensure that it is correct. After he has done that, he will examine the problem again in order to note its characteristics. He will write them on the blackboard (along with the method of solution used to correctly solve this problem). The characteristics are:

1. A decision needs to be made between two alternatives (Can he do it or can he not do it?).

2. A specialized problem could give a clue as to the choice (to be made between the two alternatives) or decide the choice.

25 min. Instructional Strategy For Problem Two

The teacher, after demonstrating how to solve problem one, will then ask the students to try problem two; to imitate the method that the instructor used to solve the previous problem. After about fifteen minutes, the teacher will take up the problem with them. He will ask the same series of questions that he asked in prob-

lem one; that is:

> What is the unknown?
> What information is given?
> Do you know a related problem?

In this problem, the instructor will attempt to have the students give the answers to the first two questions. If the students have difficulty with either or both these questions, then the teacher will tell them the answer(s).

Before they respond to the third question, the teacher will ask them to name the characteristics of the problem. He will then ask them to think of a related problem and to name its method of solution. (Since this problem is characterized as a problem involving a specialized case, whose solution is used in solving the main problem, the method of solution decided upon by the students and/or the teacher is specialization).

The instructor will then ask the question: Do you know a more specialized problem? If the students could not solve the more specialized problem themselves, then the teacher will demonstrate how to solve it. If the students could not use the special problem to help solve the original problem, then the instructor will show the students how this is done. (The more specialized problem that will be used in devising a plan and the steps that will be followed in carrying out the plan are all presented

in this thesis under the heading: "Answers to Problems From the Sixth Day").

After the plan has been carried out a a solution has been reached, the teacher will ask the questions:

> Is the result correct?
>
> What are the characteristics of this problem that could be associated with the method of solution that was utilized?

The instructor will first check the result to ensure that it is correct. After he has done that, he will ask the students again, what are the characteristics of the problem. He will then write them on the blackboard (along with the method of solution used to correctly solve this problem). The characteristics are:

1. A decision needs to be made between two alternatives (Will the first or second player consistently win the game?).

2. A specialized problem could give a clue as to the choice (to be made between the two alternatives) or decide the choice.

SIXTH DAY

Problem 1

Bob has 10 pockets and 44 silver dollars. He wants to put his dollars into his pockets so distributed that each pocket contains a different number of dollars. Can he do so?

Problem 2

In this two-person game, the players alternately place a poker chip on a circular table. The chips must not overlap and must be completely on the table; that is, no poker chip may stick out over the edge of the table. The last player to play a chip on the table is the winner. If each player makes the optimal move on his turn, will the first player or the second player be the winner?

ANSWERS TO PROBLEMS FROM THE SIXTH DAY

Problem 1

The solution to problem one has been given under the heading: "Instructional Strategy For Problem One", in this lesson.

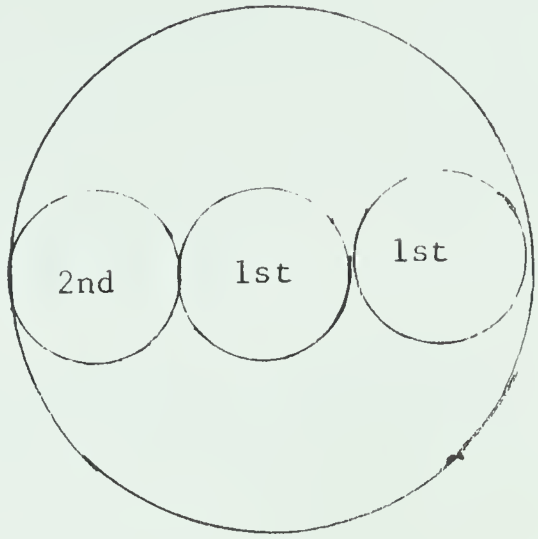
Problem 2

Special case: Consider a table that is big enough to accommodate only one poker chip (when placed in the center of the table).

In such a case, the player who goes first will be able to place the first and last poker chip on the table and will therefore be the winner.

This case suggests that, if one of the players has a forced win playing by optimal strategy for all sizes of tables, then that player is the first player.

In solving the problem, the first player initially places a poker chip in the center of the table (as in the special case) and thereafter plays chips in a symmetrically opposite position to that played by the second player.



Clearly, if the second player has any place on the table available to place a poker chip, there will still be a symmetrically opposite place on the table for the first player to place a chip, so that the first player must be the last to play a chip on the table, independent of the size of the table.

7. Lesson Seven

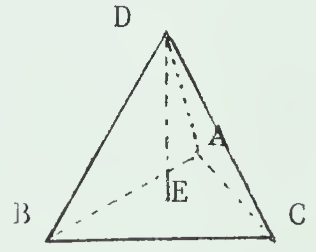
Time	Topic
	Problem Solving (Analogy)

20 min.

Instructional Strategy For Problem One

The instructor will write the problem on the blackboard and then pause for several minutes allowing the students sufficient time to understand the problem. The instructor will then begin by asking: What is the unknown? He will answer that the unknown is the altitude of the regular tetrahedron. The second question will be: What information is given? The given data are a regular tetrahedron with edge length 10 cm , and that a median of a median of a triangular base is intersected into two line segments by the other two medians, whose lengths are in the ratio 2:1.

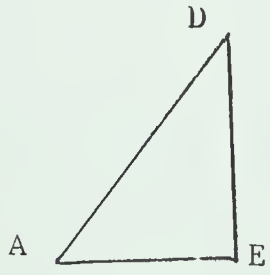
Since the unknown and the data have been delineated, it remains to devise some plan by which the teacher might proceed from the given to the unknown. The instructor will begin by drawing a diagram depicting the original situation.



He will then ask the question: Do you know related problem? The purpose of this question is to have the students discover a method of solution which has a reasonable chance of leading to the correct solution. Before responding to this question, the teacher will name the characteristics of the problem currently before him. He will then think of a related problem (with similar characteristics) and name its method of solution. Since this problem is characterized as a problem involving a three-dimensional figure, but asking for a one-dimensional answer (that is, the altitude of the regular tetrahedron), the method of solution decided upon by the teacher is analogy.

The instructor will then ask the question: Do you know an analogous problem? The answer that the teacher will give is to write an analogous problem on the blackboard. He will then proceed to solve the analogous problem.

> Given a right triangle ADE and given the length of the sides $\overline{AD}$ and $\overline{AE}$; find the length of side $\overline{DE}$.



Pythagoras' theorem states: $(\overline{AD})^2 = (\overline{AE})^2 + (\overline{DE})^2$. Therefore, $(\overline{DE}) = \sqrt{(\overline{AD})^2 - (\overline{AE})^2}$.

The procedure that he used in solving this analogous problem will be helpful in guiding him through the solution to the original problem.

<u>First</u>:

Triangle ADE is a right triangle (because $\overline{DE}$ is the altitude).

Therefore $\overline{AD}^2 = \overline{AE}^2 + \overline{DE}^2$ (Pythagoras' theorem)

or $\overline{DE} = \sqrt{\overline{AD}^2 - \overline{AE}^2}$

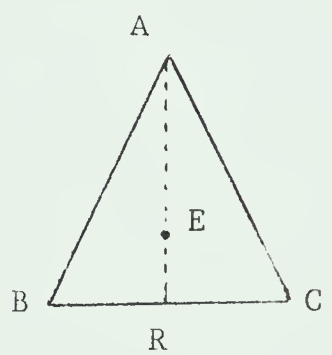
but $\overline{AD} = 10$ cm. (edge of tetrahedron)

Therefore $\overline{DE} = \sqrt{100 - AE^2}$

<u>Second</u>:

Find $\overline{AE}$.

Examine triangle ABC.



Given altitude $\overline{AR}$.

$\overline{ER} = \frac{1}{2}\overline{AE}$

Find $\overline{AR}$.

Examine right triangle ARC.

Pythagoras' theorem states:

$$\overline{AC}^2 = \overline{RC}^2 + \overline{AR}^2$$

$$10^2 = 5^2 + \overline{AR}^2$$

$$\overline{AR} = \sqrt{10^2 - 5^2}$$

$$= 5\sqrt{3} \text{ cm}$$

but $\overline{AR} = \overline{AE} + \overline{ER} = 5\sqrt{3}$ cm

$$= \overline{AE} + \tfrac{1}{2}\overline{AE} = 5\sqrt{3} \text{ cm}$$

Therefore $(3/2)(\overline{AE}) = 5\sqrt{3}$ cm

$$\overline{AE} = 10\sqrt{3}/3 \text{ cm}$$

<u>Third</u>:

$$\overline{DE} = \sqrt{100 - AE^2}$$

$$= \sqrt{100 - 100(3)/9}$$

$$= \sqrt{(900-300)/9}$$

$$= 10\sqrt{6}/3 \text{ cm}$$

$$= 8.16 \text{ cm}$$

Therefore the height of the regular tetrahedron is 8.16 cm.

After the plan has been carried out and a solution has been reached, the instructor will ask the questions:

> Is the result correct?
>
> What are characteristics of this problem that could be associated with the method of solution that was utilized?

The teacher will first check his result to ensure that it is correct. After he has done that, he will examine the problem again in order to note its characteristics. He will write them on the blackboard (along with the method of solution used to correctly solve this problem). The

characteristics are:

1. The problem involves a three-dimensional figure (a regular tetrahedron), but is asking for a one-dimensional answer (altitude).

2. Using a triangle (a two-dimensional figure) to find the altitude of a regular tetrahedron (a three-dimensional figure), appears to be a necessity. The triangle in a plane is analogous to a tetrahedron in space.

20 min.

Instructional Strategy For Problem Two

The teacher, after demonstrating how to solve problem one, will then ask the students to try problem two; to imitate the method that the instructor used to solve the previous problem. After about ten minutes, the teacher will take up the problem with them. He will begin by drawing a diagram depicting the original situation. The instructor will then ask the same series of questions that he asked in question one; that is:

What is the unknown?
What is the given information?
Do you know a related problem?

In this problem, the teacher will attempt to have the students give the answers to the first two questions. If the students have difficulty with either or both of these questions, the instructor will tell them the answer.

Before they respond to the third question, the teacher will ask them to name the characteristics of this problem. Based on these characteristics, he will then ask them to think of a related problem and to name its method of solu-

tion. Since the problem is characterized as a problem involving a three-dimensional figure, but asking for a one-dimensional answer (that is, the diagonal of the rectangular solid), the method of solution decided upon by the students and/or the teacher is analogy.

The instructor will then ask the question: Do you know an analogous problem? If the students could not both give the analogous problem and its solution themselves, the teacher will demonstrate how to solve it. If the students could not use the analogous problem to help solve the original problem, the instructor will show them how this is done. (The diagram and the analogous problem that will be sued in devising the plan and the steps followed in solving the problem are all presented in this thesis under the heading: "Answers to Problems From the Seventh Day").

After the plan has been carried out and a solution has been reached, the teacher will ask the questions:

> Is the result correct?
>
> What are the characteristics of this problem that could be associated with the method of solution that was utilized?

The instructor will first check the result to ensure that it is correct. After he has done that, he will ask the students again, what are the characteristics of the problem. He will then write them on the blackboard (along

with the method of solution used to correctly solve this problem). The characteristics are:

1. The problem involves a three-dimensional figure (a rectangular solid), but is asking for a one-dimensional answer (diagonal).

2. Using a triangle (a two-dimensional figure) to find the altitude of a rectangular solid (a three-dimensional figure), appears to be a necessity. The triangle in a plane is analogous to a tetrahedron in space.

SEVENTH DAY

Problem 1

Find the altitude of a regular tetrahedron with given edge 10 cm. (Hint: the lower end of the altitude intersects a median of the triangular base into two line segments whose lengths are in the ratio 2:1).

Problem 2

Find the diagonal of a rectangular solid of which the length, width, and height are "a" cm, "b" cm, and "c" cm respectively.

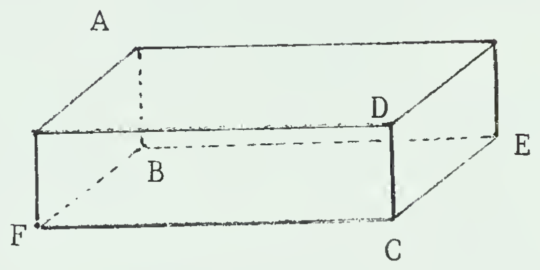
ANSWERS TO PROBLEMS FROM THE SEVENTH DAY

Problem 1

The solution to problem one has been given under the heading: "Instructional Strategy For Problem One", in this lesson.

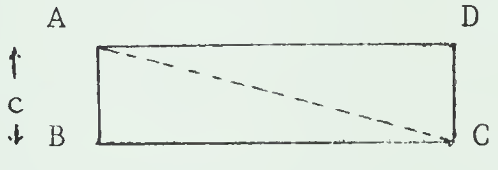
Problem 2

Draw a diagram of the rectangular solid, length "a" cm , width "b" cm , and height "c" cm.



The problem is to find the length of $\overline{AC}$ (or $\overline{BD}$).

Take the analogous problem: Given a rectangle ABCD, find the length of the diagonal $\overline{AC}$ (or $\overline{BD}$).



First:

Pythagoras' theorem states:

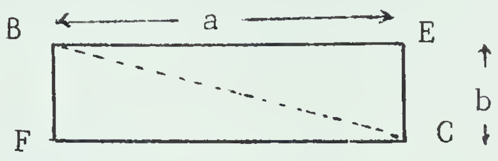
$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 \text{ (Pythagoras)}$$

$$\overline{AC} = \sqrt{\overline{AB}^2 + \overline{BC}^2}$$

$$= \sqrt{c^2 + \overline{BC}^2}$$

<u>Second</u>:

Take the rectangular base of the solid.



$$\overline{BC}^2 = \overline{BE}^2 + \overline{EC}^2$$
$$= a^2 + b^2$$

<u>Third</u>:

Therefore $\overline{AC} = \sqrt{c^2 + \overline{BC}^2}$

$$= \sqrt{c^2 + (a^2+b^2)}$$

Thus the length of the diagonal of the rectangular solid is $\sqrt{a^2 + b^2 + c^2}$ cm.

8. Lesson Eight

Time — Topic

Problem Solving (Decomposing and Recombining)

Introduction

The teacher will hand out a sheet of paper, to each student, containing five problems (listed in this lesson under the heading: "Eighth Day"). The first two problems will be solved in class during this period. The last three problems will be assigned for homework and taken up in class next day.

20 min. — Instructional Strategy For Problem One

The teacher will read the first problem aloud and then pause for several minutes allowing the students sufficient time to understand the problem. The instructor will then begin by asking: What is the unknown? He will answer that the unknown is the salary for one month, and he will write this on the blackboard. The second question will be: What information is given? The given data are that 18% of the salary is for rent and that the rent is $45 per month.

Since the unknown and the data have been delineated, it remains to devise some plan by which the teacher might

proceed from the given data to the unknown. He will then ask himself the question: Do you know a related problem? The purpose of this question is to have the students discover a method of solution which has a reasonable chance of leading to the correct solution. Before responding to this question, the teacher will name the characteristics of the problem currently before him. He will then think of a related problem (with similar characteristics) and name its method solution. Since this problem is characteristized as a problem in which a mathematics sentence can be established, the method of solution decided upon by the teacher is decomposing and recombining.

The instructor will then ask himself the question: Can you restate the problem? The answer that the teacher will give is to restate the problem in such a way that he progresses closer to the solution to this problem. In this case, the instructor will set up a mathematics sentence, expressing in mathematical symbols, a condition that is stated in words. The equivalent mathematics sentence is: $(.18)(x) = 45$, where x is the salary for one month. He will then solve the equation for x and thus find the value of the unknown number.

Let \$x be the salary per month.

ORIGINAL PROBLEM	ALGEBRAIC TRANSLATION
A man spends 18% of his monthly salary for rent.	.18
If his rent is \$45 a month,	45
what is his salary for a month?	x

$(.18)(x) = 45$ or $(18)/(100) = (45)/(x)$

$x = 45/(.18)$

$= 250$

Therefore his salary for a month is \$250.

After the plan has been carried out and a solution has been reached, the teacher will ask the questions:

> Is the result correct?
>
> What are the characteristics of this problem that could be associated with the method of solution that was utilized?

The instructor will first check his result to ensure that it is correct. After he has done that, he will examine the problem again in order to note its characteristics. He will write them on the blackboard (along with the method of solution used to correctly solve this problem). The characteristics are:

1. A problem in which one comparison is made between the rent and the salary.

2. A problem that is easily translated into a mathematics sentence.

20 min.

Instructional Strategy For Problem Two

The teacher, after demonstrating how to solve problem one, will then ask the students to try problem two; to imitate the method that the instructor used to solve the previous problem. After about ten minutes the teacher will take up the problem with them. The instructor will ask the same series of questions that he asked for problem one, that is:

What is the unknown?
What information is given?
Do you know a related problem?

In this problem, the teacher will attempt to have the students give the answers to the first two questions. If the students have difficulty with either or both of these questions, the instructor will tell them the answers.

Before they respond to the third question, the teacher will ask them to name the characteristics of this problem. Based on these characteristics, he will then ask them to think of a related problem and to name its method of solution. Since this problem was characterized by the students as a problem that involves two unknowns (Anne's and Jane's

current ages) and two other facts (sufficient to set up two equations), the method of solution decided upon by the students and/or the teacher is decomposing and re-combining.

The instructor will then ask the question: Can you restate the problem? In answer to this question, the teacher will expect the students to answer by giving the translation of the English sentences of the problem into mathematics sentences. Once again, if the students are unable to give him the correct mathematics sentence, then the instructor will tell them what is the correct answer. If however, the students got the correct mathematics sentence but failed to carry out the computation correctly, then the instructor will show them how to solve the equation correctly. (The mathematics sentences that will be used in devising the plan and the steps following this sentence that will be used in carrying out the plan are all presented in this thesis under the heading: "Answers to Problems From the Eighth Day").

After the plan has been carried out and a solution has been reached, the teacher will ask the questions:

> Is the result correct?
>
> What are the characteristics of this problem that could be associated with the method of solution that was utilized?

The instructor will first check the result to ensure that

it is correct. After he has done that, he will ask the students again, what are the characteristics of the problem. He will then write them on the blackboard (along with the method of solution used to correctly solve this problem). The characteristics are:

1. A problem in which two comparisons are made between Anne's and Jane's ages; thus two equations can be established in order to generate the solution.

2. A problem which is easily translated into two mathematics sentences.

EIGHTH DAY

Problem 1

A man spends 18% of his monthly income for rent. If his rent is $45 a month, what is his salary for a month?

Problem 2

Anne is 7 years older than Jane. One year ago, she was twice as old as Jane. How old is each now?

Problem 3

The difference between two numbers is 10. If twice the larger is subtracted from five times the smaller, the remainder is 22. What are the numbers?

Problem 4

How many kilograms of candy worth $1.50 per kilogram and how many kilograms of candy worth $2.70 per kilogram must be used to make a 225 kilogram mixture which is worth $2.28 per kilogram?

Problem 5

John has a dream about airplane pilots and sports cars. When all the airplane pilots in John's dream got into sports cars, there was one pilot per car, and seven cars were still empty. Then one half of the pilots departed by jet for Hawaii. After that, when all of the remaining pilots got into sports cars, there was one pilot per car, but twenty-nine cars were empty. How many sports cars were there in John's dream?

ANSWERS TO PROBLEMS FROM THE EIGHTH DAY

Problem 1

The solution to problem one has been given under the heading: "Instructional Strategy For Problem One", in this lesson.

Problem 2

Let x years be Anne's age now.

Let y years be Jane's age now.

ORIGINAL PROBLEM	ALGEBRAIC TRANSLATION
Anne is 7 years older than Jane.	$x = 7+y$
One year ago, she was twice as old as Jane.	$x-1 = 2(y-1)$
How old is each now?	x,y ?

$x = 7+y$ (call this equation #1)

$x-1 = 2(y-1)$ (call this equation #2)

Substitute equation #1 into equation #2.

$$(7+y) - 1 = 2(y-1)$$

$$y+6 = 2y-2$$

$$y = 8 \text{ years.}$$

Solve for x in equation #1.

$$x = 7+8$$

$$15 \text{ years}$$

Check in equation #2.

$$x-1 = 2(y-1)$$

$$15-1 = 2(8-1)$$

$$14 = 14$$

Therefore Anne is 15 years old and Jane is 8 years old.

9. <u>Lesson Nine</u>

<u>Time</u> <u>Topic</u>

Problem Solving (Decomposing and Recombining)

<u>Instructional Strategy For Problems Three, Four, and Five From Lesson Eight</u>

The teacher will ask three students to come to the blackboard and write out their solutions, one pupil to each problem. After the three solutions have been written on the blackboard, the students will be asked to take their seats. The instructor will then ask the class if any corrections need to be made with regard to the first two questions (that is, "What is the unknown?" and "What is the given information?") in each solution. If there are any errors, and if the class has difficulty in locating them, the teacher will make the corrections himself. The instructor will then examine each of the three solutions on the blackboard in order to see how they answered the question: Do you know a related problem? He will solicit responses from the class as to the characteristics of the three problems.

<u>For Problem Three</u>:

Since this problem was characterized by the students (or teacher) as a problem involving two comparisons be-

tween two different numbers (sufficient information to set up two equations), the method of solution decided upon by the students and/or the teacher is decomposing and recombining.

For Problem Four:

Since this problem was characterized by the students (or teacher) as a problem involving two comparisons between two kinds of candy (sufficient information to set up two equations), the method of solution decided upon by the students and/or the teacher is decomposing and recombining.

For Problem Five:

Since this problem was characterized by the students (or teacher) as a problem involving two comparisons between pilots and sports cars (sufficient information to set up two equations), the method of solution decided upon by the students and/or the teacher is decomposing and recombining.

The instructor will then examine the three solutions to see how the three students answered the question: Can you restate the problem? In answering this question, the teacher will expect the three students to have translated the English sentences of the problems into mathematics sentences. Once again, if the students were unable to write the correct mathematics sentences, then the instructor will tell them the correct answers. If however,

the students got the correct mathematics sentences but failed to carry out the computations correctly, then the instructor will show them how to solve the equations correctly. (The mathematics sentences that will be used in devising the plan and the steps following these sentences that will be used in carrying out the plan are all presented in this thesis under the heading: "Answers to Problems From the Ninth Day").

After the plan has been carried out and a solution has been reached, the teacher will check the blackboard to note their responses to the questions:

> Is the result correct?
>
> What are the characteristics of this problem that could be associated with the method of solution that was utilized?

The characteristics of problem three are:

1. A problem in which two comparisons are made between two numbers; thus two equations can be established in order to generate the solution.

2. A problem which is easily translated into two mathematics sentences.

The characteristics of problem four are:

1. A problem in which two comparisons are made between two kinds of candy; thus two equations can be established in order to generate the solution.

2. A problem which is easily translated into two mathematics sentences.

The characteristics of problem five are:

1. A problem in which two comparisons are made between pilots and sports cars; thus two equations can be established in order to generate the solution.

2. A problem which is easily translated into two mathematics sentences.

NINTH DAY

The three problems that were discussed in this lesson are all presented in Lesson Eight under the heading: "Eighth Day".

ANSWERS TO PROBLEMS FROM THE NINTH DAY

Problem 3

Let x be the larger number.

Let y be the smaller number.

ORIGINAL PROBLEM	ALGEBRAIC TRANSLATION
The difference between 2 numbers is 10.	$x-y = 10$
If twice the larger is subtracted from 5 times the smaller,	$5y-2x$
the remainder is 22.	22
What are the numbers?	x,y ?

$x-y = 10$ (call this equation #1)

$-2x+5y = 22$ (call this equation #2)

Multiply equation #1 by 2.

$2x-2y = 20$ (call this equation #3)

Add equation #2 and equation #3.

$3y = 42$

$y = 14$

Solve for x in equation #1.

$x-14 = 10$

$x = 24$

Check in equation #2.

$$-2(24)+5(14) = 22$$

$$22 = 22$$

Therefore the numbers are 14 and 24.

Problem 4

Let x kilograms be amount candy worth \$1.50/kg.

and \$1.5x be the value of candy worth \$1.50/kg.

Let y kilograms be amount candy worth \$2.70/kg.

and \$2.7y be the value of candy worth \$2.70/kg.

ORIGINAL PROBLEM	ALGEBRAIC TRANSLATION
How many kilo-grams of candy	x kg.
worth \$1.50/kg.	\$1.50/kg.
and how many kilo-grams of candy	y kg.
worth \$2.70/kg.	\$2.70/kg.
must be used to make a 225 kg. mixture	225 kg.
which is worth \$2.28/kg.?	\$2.28/kg.

$$x+y = 225 \quad \text{(call this equation \#1)}$$

$$1.5x+2.7y = (225)(2.28)$$

$$= 513 \quad \text{(call this equation \#2)}$$

Multiply equation #2 by 10.

$$15x+27y = 5130 \quad \text{(call this equation \#3)}$$

Multiply equation #1 by 15.

$$15x+15y = 3375 \quad \text{(call this equation \#4)}$$

Subtract equation #4 from equation #3.

$$12y = 1755$$

$$y = 146.25 \text{ kg.}$$

Solve for x in equation #1.

$$x+146.25 = 225$$

$$x = 78.75 \text{ kg.}$$

Check in equation #2.

$$(1.5)(78.75)+(2.7)(146.25) = 513$$

$$513 = 513$$

Therefore there are 78.75 kilograms worth \$1.50/kg. and 146.25 kilograms worth \$2.70/kg.

Problem 5

Let x be the number of sports cars in John's dream.

Let y be the number of pilots in John's dream.

ORIGINAL PROBLEM	ALGEBRAIC TRANSLATION
John has a dream about pilots and sports cars.	x, y
When all the airplane pilots in John's dream got into sports cars, there was one pilot per car and seven cars were still empty.	$x-y = 7$
Then one half of the pilots departed by jet for Hawaii.	$y/2$
After that, when the remaining pilots	$y/2$
got into the sports cars, there was one pilot per car, but 29 cars were empty.	$x-(y/2) = 29$
How many sports cars were there in John's dream?	x ?

$$x-y = 7 \quad \text{(call this equation \#1)}$$

$$x-(y/2) = 29 \quad \text{(call this equation \#2)}$$

Subtract equation #1 from equation #2.

$$y/2 = 22$$

$$y = 44$$

Solve for x in equation #1.

$$x-y = 7$$

$$x-44 = 7$$

$$x = 51$$

Check in equation #2.

$$x-(y/2) = 29$$

$$51-(44/2) = 29$$

$$29 = 29$$

Therefore there were 51 sports cars in John's dream.

10. Lesson Ten

(a) Lesson Plan

Time	Topic
	Five-Method Comprehension Test
50 min.	Administration of the Test The teacher will hand out the tests and in doing so, he will tell the students to fill in their name on the first page of the test. (The Five-Method Comprehension Test is contained in this thesis in Appendix B).

(b) Scoring Procedure For the Five-Method Comprehension Test

Problem 1 (10 marks)

Unknown? (1 mark)

- number of resultant triangles

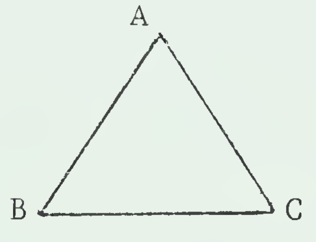
Given? (1 mark)

- triangle ABC
- 25 lines drawn through vertex A and opposite side

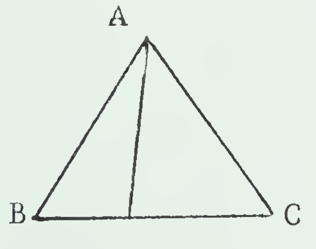
Related Problem?

- characteristics (2 marks)
  - number of lines is large; number of resultant triangles is very large.
  - drawing a diagram depicting the problem and counting the number of resultant triangles is an extremely difficult task.
- method of solution - given as generalization
- Related problems? (2 marks)

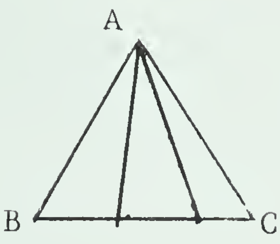
Case #1 -



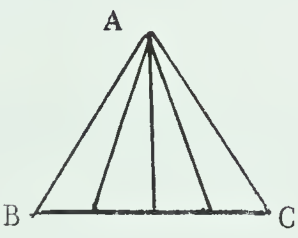
Case #2 -



Case #3 -



Case #4 -



Number of Lines	Number of Triangles
0	1
1	3
2	6
3	10
4	15
5	21
6	28
7	36
8	45
9	55
10	66
11	78
12	91
13	105
14	120
15	136
16	153
17	171
18	190
19	210
20	231
21	253
22	276
23	300
24	325
25	351

(1 mark for setting up the table with titles and first 3 or 4 rows of data)

(1 mark for recognition of pattern; shown by at least a fifth row in the table)

(2 marks for the correct answer)

<u>Problem 2</u> (10 marks)

Unknown? (1 mark)

- how to get 5 quarts of water

Given? (1 mark)

- 3-quart and 7-quart jar

- infinite supply of water

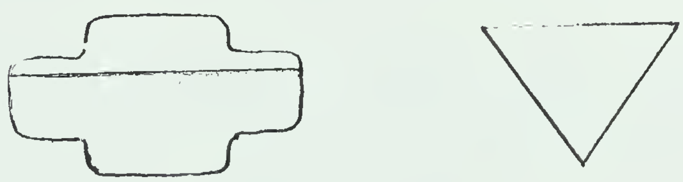
Related Problem?

- characteristics - outcome is known (5 quarts of water)
  (2 marks) - trial-and-error beginning

- method of solution - given as working backward

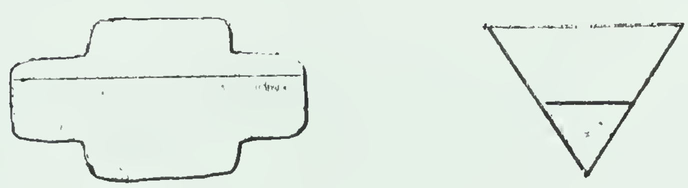
- Restated problem? (1 mark)

  Given 3-quart and 7-quart jar and infinite water supply. Also given 5 quarts of water in 7-quart jar. How did they get there?



(1 mark)

Case #1 -
(Diagrams depict problem as solved)



(1 mark)

Case #2 -
(Diagrams depict pouring out 2 quarts of water into 3-quart jar in which there is already 1 quart of water)

 (1 mark)

Case #3 -
(Diagram depicts 1 quart of water in 3-quart jar. Filled the 7-quart jar, poured out equivalent of 2 3-quart jars, and poured the remaining 1 quart into the 3-quart jar).

Therefore the solution is: First, get 1 quart in 3-quart jar; second, pour 2 quarts from a full 7-quart jar; third, there are 5 quarts remaining in the 7-quart jar. (2 marks)

<u>Problem 3</u> (10 marks)

Unknown? (1 mark)

- Possibility of placing a different number of stamps onto each of the 12 pages, an odd number of stamps on the odd-numbered pages and an even number of stamps on the even-numbered pages.

Given? (1 mark)

- 67 stamps
- 12 pages
- different number of stamps on each page
- odd-numbered pages contain an odd number of stamps
- even-numbered pages contain an even number of stamps

Related Problem?

- characteristics (2 marks) - decision needs to be made between two alternatives (can he do it or can he not do it?)
  - specialized problem could give a clue as to the choice (between alternatives) or decide the choice

- method of solution - given as specialization

- Specialized problem? (1 mark)

  What is the minimum number of stamps necessary in order to satisfy the requirements of the problem.

1st page contains 1 stamp	2nd page contains 0 stamps
2nd page contains 3 stamps	4th page contains 2 stamps
.	.
.	. (1½ marks)
.	.
11th page contains 11 stamps	12th page contains 10 stamps

The minimum number of stamps = (1+3+...+11)+(0+2+...+10)
= 36+30
= 66 (1½ marks)

Since John has 67 stamps he can put a different number onto each of the 12 pages. (2 marks for conclusion)

<u>Problem 4</u> (10 marks)

Unknown? (1 mark)

- diagonal of the rectangular solid

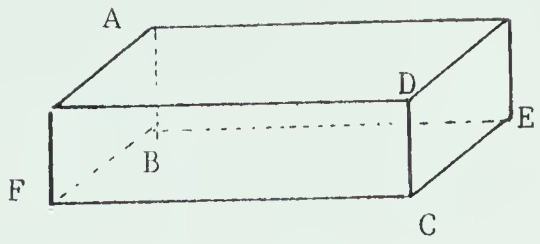
Given? (1 mark)

- rectangular solid with length 12 cm., width 4 cm., and height 3 cm.

Related Problem?

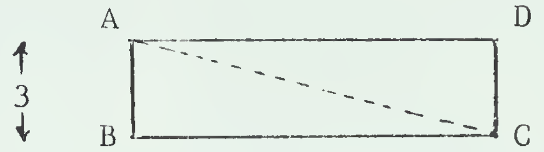
- characteristics (2 marks)
  - given a 3-dimensional figure; wanted a 1-dimensional answer.
  - the 3-dimensional figure can be easily simplified into a 2-dimensional figure.

- method of solution - given as analogy



Analogous problem? (1 mark)

Given a rectangle ABCD, find the length of the diagonal $\overline{AC}$ (or $\overline{BD}$).

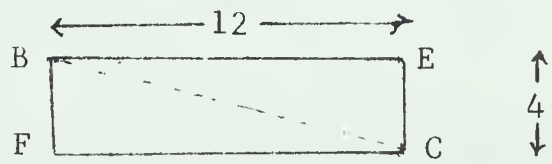


First: $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$ (Pythagoras)

$$\overline{AC} = \sqrt{\overline{AB}^2 + \overline{BC}^2}$$

$$= \sqrt{3^2 + \overline{BC}^2} \qquad \text{(2 marks)}$$

Second: Take the rectangular base of the solid



$$\overline{BC}^2 = \overline{BE}^2 + \overline{EC}^2$$

$$= 12^2 + 4^2 \qquad \text{(1 mark)}$$

Third: Therefore $\overline{AC} = \sqrt{3^2 + \overline{BC}^2}$

$$= \sqrt{3^2 + (12^2+4^2)} \qquad \text{(2 marks)}$$

Thus the length of the diagonal of the rectangular solid is $\sqrt{3^2+12^2+4^2} = 13$ cm.

Problem 5 (10 marks)

Unknown? (1 mark)

- the smaller and larger numbers

Given? (1 mark)

- difference between them is 21
- 32 times smaller minus 4 times the larger equals 28

Related Problem?

- characteristics (2 marks) - two comparisons are made between two numbers
  - easily translated into two mathematics sentences

- method of solution - given as decomposing and recombining

- Restated problem? (1 mark for naming unknowns and 1 mark for table)

Let x be the larger number.

Let y be the smaller number.

ORIGINAL PROBLEM	ALGEBRAIC TRANSLATION
The difference between 2 numbers is 21.	$x-y = 21$
If 4 times the larger is subtracted from 32 times the smaller,	$32y-4x$
the remainder is 28.	28
What are the numbers?	x,y ?

$x-y = 21$ (call this equation #1)

$-4x+32y = 28$ (divide by 4)

$-x+8y = 7$ (call this equation #2)

Add equation #1 and equation #2.

$7y = 28$

$y = 4$

Solve for x in equation #1.

$x-4 = 21$

$x = 25$

Check in equation #2.

$-(25)+8(4) = 7$

$7 = 7$

Therefore the numbers are 4 and 25.

(1 mark for equation #1)

(1 mark for equation #2)

(2 marks fore the answer)

11. Lesson Eleven

Time	Topics
	1. The Practical Application of Each Method of Solution
	2. The Characteristics Associated With Each Method of Solution
	3. The Transition From Lessons 1-10 to Lessons 11-13
	4. The Return and Discussion of Five-Method Comprehension Test

5 min.

The teacher will begin this lesson by reviewing the concepts that the students should have learned in the previous nine lessons. He will mention the questioning sequence that characterizes the approach that should be taken in solving every problem. That is:

1st: Unknown?

2nd: Given?

3rd: Related problem?

- name the characteristics of the problem
- name the associated method of solution

4th: Method implemented?

5th: Check?

6th: Write the characteristics and method of solution of this problem.

10 min. He will name the five methods of solution that the students have taken in lessons 2-9, in order of presentation. They are:

1. Generalization
2. Working Backward
3. Specialization
4. Analogy
5. Decomposing and Recombining

After he has listed these five solution methods, the instructor will discuss examples of their usage.

Generalization - Scientific field (Gathering of data in order to discover the pattern and to develop the formula which expresses the generalized case.)
- Research field (Gathering data from a sample to generalize to a population.)

Working Backward - Game situations (Observer sees the result and asks: "How did he do that?")
- Hobbies (In trying to duplicate some finished project, the individual works backward from the completed project.)
- Generally, in problems in which the solution is known and the procedure is unknown.

Specialization - Field of mathematics: proving or disproving theorems.

Analogy - Field of mathematics: geometry

- Science (In situations in which the concept is abstract and therefore difficult to visualize. For example, corn that is popping over a hot fire is analogous to molecular motion).

Decomposing and Recombining - Field of mathematics (It is the major method of solution that is used. It involves building systems of equations.

10 min. The instructor, after mentioning these practical implications, will associate each of the solution methods with their appropriate characteristics. He will write the chart, as given below, on the blackboard and ask the students to copy it into their notebooks.

METHOD OF SOLUTION	CHARACTERISTICS
Generalization	- number of "lines" is large; number of resultant "triangles" is very large. - drawing a diagram depicting the problem and counting the number of resultant "triangles" is an extremely difficult task.
Working Backward	- outcome is known ("6 quarts of water") - trial-and-error beginning

Specialization	- decision needs to be made between two alternatives ("Can he do it or can he not do it?"). - specialized problem could give a clue as to the choice or decide the choice.
Analogy	- given a 3-dimensional figure; wanted a 1-dimensional answer. - the 3-dimensional figure can be easily simplified into a 2-dimensional figure.
Decomposing and Recombining	- "two" comparisons are made between "two" "numbers". - easily translated into "two" mathematics sentences.

The importance of these associations will become evident in the next three lessons. That is, in the upcoming three lessons, the students will be presented with problems whose method of solution is unknown. They will be required to examine the problem, characterize the problem, associate these characteristics with their appropriate method of solution, and then solve the problem.

15 min.

Finally, the teacher will give back the students' answer sheets to the Five-Method Comprehension Test. He will use lesson ten, subsection (b) entitled "Scoring Procedures For the Five-Method Comprehension Test" in

his discussion of the test. (Results of the pilot testing of this instrument indicate that special attention should be given to discussing problems three and four, whose solution methods are specialization and analogy respectively).

At the end of the period the teacher will hand out three sheets of paper to each pupil, containing the seven problems that the students will be solving in the next three lessons. (These problems are contained in Lesson Twelve under the heading: "Twelfth Day"). He will ask the students to read the first three problems in preparation for the next day's class.

12. Lesson Twelve

Time	Topic
	Problem Solving (Mixed Problems)

Introduction

The teacher will tell the students that for the next three periods, they will be given a variety of problems to solve. The major skill that the problem solver will be expected to develop during the next three lessons is how to correctly characterize a problem; following from that, to choose the appropriate method of solution, and to utilize it correctly in solving a problem.

40 min. Instructional Strategy For Problems One, Two, and Three

The instructor will allow the students about twenty-five minutes to solve the first three problems. He will then ask three students to come to the blackboard and write out their solutions, one pupil to each problem. After the three solutions have been written on the blackboard, the students will be asked to take their seats. The instructor will then ask the class if any corrections need to be made with regard to the first two questions (that is, "What is the unknown?" and "What is the given

information?") in each solution. If there are any errors, and if the class has difficulty in locating them, the teacher will make the corrections himself. The instructor will then examine each of the three solutions on the blackboard in order to see how they answered the question: Do you know a related problem? He will solicit responses from the class as to the characteristics of the three problems.

<u>For Problem One</u>:

Since this problem was characterized by the students (or teacher) as a problem involving an equation of the fourth degree ($4x^4$-$33x^2 27$=0) but asking for an answer expressed in the form of a linear equation, the method of solution decided upon by the students and/or the teacher is analogy.

<u>For Problem Two</u>:

Since this problem was characterized by the students (or teacher) as a problem involving pattern development, the method of solution decided upon by the students and/or the teacher is generalization.

<u>For Problem Three</u>:

Since this problem was characterized by the students (or teacher) as a problem involving two comparisons between two amounts of money, the method of solution decided upon by the students and/or the teacher is decomposing and recombining.

The instructor will then examine the three solutions in order to ascertain whether or not the three students correctly implemented the respective solution methods. (The solutions to these three problems are presented in this thesis under the heading: "Answers to Problems From the Twelfth Day").

After the plan has been carried out and a solution has been reached, the teacher will check the blackboard to note their responses to the questions:

> Is the result correct?
>
> What are the characteristics of this problem that could be associated with the method of solution that was utilized?

The characteristics of problem one are:

1. Given a problem involving a quartic equation ($4x^4-33x^2-27=0$); wanted a linear equation for the answer.

2. The quartic euqation can easily be simplified into a quadratic equation ($4y^2-33y-27=0$).

The characteristics of problem two are:

1. The number of numbers is large; the sum is very large.

2. Adding up the one hundred numbers is an extremely tedious task.

The characteristics of problem three are:

1. Two comparisons are made between two amounts of money.

2. The problem is easily translated into two mathematics sentences.

TWELFTH DAY

Problem 1

Bill and Doug are preparing for an international math competition, to be administered the following week. They are asking each other questions. One problem that Bill gave Doug to solve was real challenging. It stated: "A boy asked his father how much money he (the father) had in his wallet. The father's reply was in the form of a riddle. He said: the number of dollars that I have is multiplied by itself four times and the result multiplied by four. From this amount, thirty-three times the square of the original amount is subtracted. The result is $27. How many dollars did he have in his wallet?"

Problem 2

Find the sum of the first one hundred natural numbers.

Problem 3

A lady has one sum of money invested at 5% per annum and a second sum, $1500 larger than the first, invested at 6% per annum. Her total income from these sums is $200 (after one year of investment). How much has she invested at each rate?

Problem 4

On an infinitely extended checkerboard, one is given three black checkers and two white checkers initially placed in immediately adjacent squares on a single row, proceeding

from left to right, as shown in the figure below: black (B), white (W), black, white, black. The problem is to transform this arrangement of alternating black and white checkers into an arrangement in which all three black checkers are on the left and both white checkers are on the right (BBBWW), with all checkers being in adjacent squares and in the same row (see the figure below). The allowable operation is to move two adjacent checkers at a time, one of which must be a black checker and one a white checker. During a move, the two checkers being moved must remain together at all times, with no reversal of their left-to-right order. You are permitted to move a white-black or black-white pair of checkers to any adjacent pair of unoccupied squares along the same line. Note that there is no need to keep the checkers that are not being moved in immediately adjacent squares at any time. That is, there may be unoccupied squares between checkers at various stages between the givens and the goal. Also note that the five checkers in the goal state need not occupy the same five squares on the checkerboard as they did in the given state. They may occupy any immediately adjacent five squares in the same row.

GIVEN STATE GOAL STATE

Problem 5

Find the number of edges of a regular 16-faced polyhedron. (Hint: this polyhedron has 10 vertices).

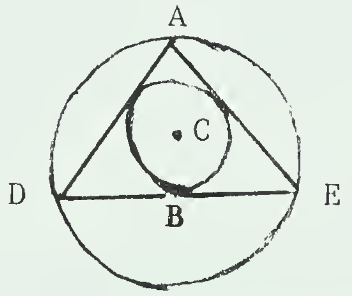
Problem 6

A farmer has hens and rabbits. These animals have 50 heads and 140 feet. How many hens and how many rabbits has the farmer?

Problem 7

In a triangle, let r be the radius of the inscribed circle, R the radius of the circumscribed circle, and H the lorgest altitude. Then $r+R \lesseqgtr H$. Prove or disprove this theorem.

Hints: 1.



Triangle ADE is any type of triangle.

2. $\overline{AB}$ is the altitude of the triangle ADE. If $\overline{AB}$ is also the median of triangle ADE, then point C intersects the median of the triangular base into two line segments whose lengths are in the ratio 2:1. In this instance, triangle ADE is an equilateral triangle.

ANSWERS TO PROBLEMS FROM THE TWELFTH DAY

Problem 1

The equation depicting the original situation is given by:

$$4x^4 - 33x^2 = 27 \text{ or}$$

$$4x^4 - 33x^2 - 27 = 0$$

where x is the number of dollars in the father's wallet.

The analogous problem is: $4y^2 - 33y - 27 = 0$ where $y = x^2$.

First:

Solve the analogous problem.

$$4y^2 - 33y - 27 = 0$$

$$(4y+3)(y-9) = 0$$

$$y = -3/4,\ 9$$

Second:

Use the result derived from the analogous problem to solve the original problem.

If $y = -3/4$

$x^2 = -3/4$

$x =$ an inadmissable value

If $y = 9$

$x^2 = 3$

$x = 3, -3$

$x = -3$ is inadmissable

Therefore, there are $3 in the father's wallet.

<u>Problem 2</u>

Case #1 -	$1 = 1$	Given the first natural number, the sum is itself.
Case #2 -	$1+2 = 3$ $= (1.5)(2)$	Given the first two natural numbers, their sum is equal to their average times the number of terms (2).
Case #3 -	$1+2+3 = 6$ $= (2)(3)$	Given the first three natural numbers, their sum is equal to their average times the number of terms.
Case #4 -	$1+2+3+4 = 10$ $= (2.5)(4)$	Given the first four natural numbers, their sum is equal to their average times the number of terms.

Number of Natural Nos.	Sum of the Natural Nos.
1	1
2	(1.5)(2)=3
3	(2)(3)=6
4	(2.5)(4)=10
.	
.	
.	
100	(50.5)(100)=5050

Therefore, given the first 100 natural numbers, their sum is 5050.

However, ideally the problem solver would only have to calculate the resulting sum using the first 1,2,3, and 4 natural numbers. From there, through a process of judicious guessing (that is, through what Polya called "trial-and-error" and the "bright idea"), the problem solver could derive the formula $n(n+1)/2$, that expresses the sum, given the first n natural numbers.

<u>Problem 3</u>

Let \$x be the amount invested at 5% per annum.

Let \$y be the amount invested at 6% per annum.

ORIGINAL PROBLEM	ALGEBRAIC TRANSLATION
A lady has one sum of money invested at 5%	\$x at 5%
and a second sum,	\$y
\$1500 larger than the first,	y = 1500+x
invested at 6%.	6%
Her total income from these sums is \$200 (after one year).	Interest=\$200 Time=1 year
How much has she invested at each rate?	x, y ?

$I_1 + I_2 = \$200$

$I_1 = prt$

$= (x)(.05)(1)$

$= .05x$

$I_2 = prt$

$= (y)(.06)(1)$

$= .06y$

Therefore $(.05x) + (.06y) = 200$

or $5x + 6y = 20000$ (call this equation #1)

And $y = 1500 + x$ (call this equation #2)

Substitute equation #2 into equation #1.

$$5x + 6(1500+x) = 20000$$

$$11x + 9000 = 20000$$

$$x = 1000$$

Solve for y in equation #1.

$$y = 1500 + x$$

$$= 1500 + 1000$$

$$= 2500$$

Check both solutions in equation #2.

$$5x + 6y = 20000$$

$$5(1000)+6(2500) = 20000$$

$$20000 = 20000$$

Therefore the amount invested at 5% was \$1000 and the amount invested at 6% was \$2500.

13. Lesson Thirteen

Time	Topic
	Problem Solving (Mixed Problems)

10 min. **Problem Three From Lesson Twelve**

If there had been insufficient time to complete problem three the previous day, it will be taken up first in this lesson. The solution to this problem will be given by the teacher using the overhead projector and prepared transparencies; it is given in Lesson Twelve under the heading: "Answers to Problems From the Twelfth Day".

30 min. **Instructional Strategy For Problems Four and Five**

The instructor will allow the students about twenty minutes to solve today's problems. He will then ask two students to come to the blackboard and write out their solutions, one pupil to each problem. After the two solutions have been written on the blackboard, the students will be asked to take their seats. The instructor will then ask the class if any corrections need to be made with regard to the first two questions (that is "What is the unknown?" and "What is the given information?") in each solution. If there are any errors, and if the class has

difficulty in locating them, the teacher will make the corrections himself. The instructor will then examine each of the two solutions on the blackboard in order to see how they answered the question: Do you know a related problem? He will solicit responses from the class as to the characteristics of the two problems.

For Problem Four:

Since this problem was characterized by the students (or teacher) as a problem in which the outcome is known (the final arrangement, BBBWW, of the five checkers), the method of solution decided upon by the students and/or the teacher is working backward.

For Problem Five:

Since this problem was characterized by the students (or teacher) as a problem involving pattern development, the method of solution decided upon by the students and/or the teacher is generalization.

The instructor will then examine the two solutions in order to ascertain whether or not the two students correctly implemented the respective solution methods. (The solutions to these two problems are presented in this thesis under the heading: "Answers to Problems From the Thirteenth Day").

After the plan has been carried out and a solution has been reached, the teacher will check the blackboard

to note their responses to the questions:

> Is the result correct?
>
> What are the characteristics of this problem that could be associated with the method of solution that was utilized?

The characteristics of problem four are:

1. The outcome is known (the final arrangement, BBBWW, of the five checkers).

2. A trial-and-error beginning.

The characteristics of problem five are:

1. The number of faces of the polyhedron is large; the number of edges is larger.

2. Drawing a diagram depicting the problem, and counting the number of edges of the polyhedron is an extremely difficult task.

THIRTEENTH DAY

The two problems that were discussed in this lesson are presented in Lesson Twelve under the heading: "Twelfth Day".

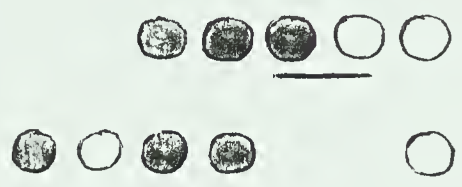
ANSWERS TO PROBLEMS FROM THE THIRTEENTH DAY

Problem 4

First: The restated problem is: Given the original conditions in the problem; also given that the player has reached the goal state; how did he get there?

Second: Start at the end of the solution and work backwards.

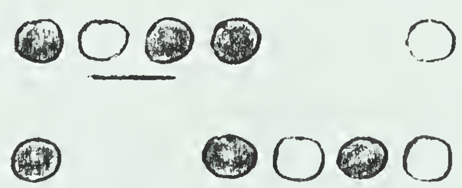
Obeying the rules, as laid down in the problem, there is only one pair of checkers that can be moved.



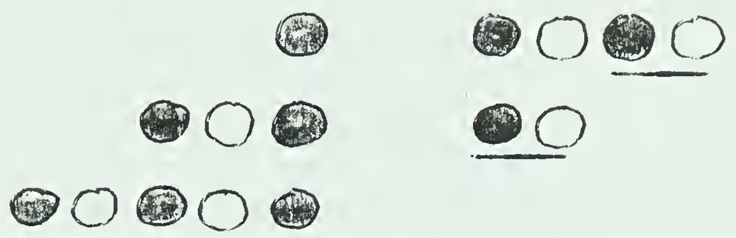
To move the pair to the left of the first black checker, establishes a BW combination on the left side (which is how the given initial state is arranged).

Third: Since the player has just moved the BW combination to the left side, he will not likely move them again on this move.

Therefore, there is only one pair of checkers that can legitimately be moved and only one reasonable place to put them (the other two possibilities would terminate the game).



Fourth: Move the two BW pairs to the left side, one pair at a time.

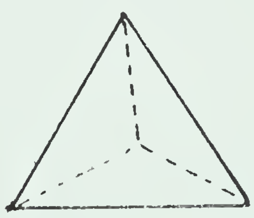


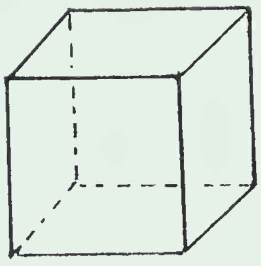
Fifth: The optimal solution is summarized in the chart below:

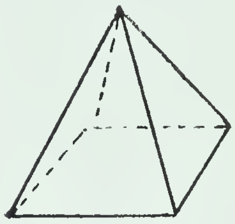
STATE	CHECKER ARRANGEMENT	ARRANGEMENT NUMBER
Goal State		
	B B B W W	1
	B W B B W	2
	B B W B W	3
	B W B B W	4
Initial State	B W B W B	5

Therefore in progressing through the game, the player starts in the initial state (arrangement #5 in the above chart) and makes the necessary moves to arrive at arrangement #4, #3, #2, and finally at #1 (the goal state).

Problem 5

Case #1 -  Given a tetrahedron, it has 4 faces, 4 vertices, and 6 edges.

Case #2 -  Given a cube, it has 6 faces, 8 vertices, and 12 edges.

Case #3 -  Given a pyramid, it has 5 faces, 5 vertices and 8 edges.

NAME OF SOLID	NUMBER FACES	NUMBER VERTICES	NUMBER EDGES
Tetrahedron	4	4	6
Cube	6	8	12
Pyramid	5	5	8
⋮	⋮	⋮	⋮
16-Faced Polyhedron	16	10	24

By examining the corresponding numbers in columns two and three in the above table, a pattern can be found. This pattern can be summarized by the formula $E = F+V-2$, where E = the number of edges, F = the number of faces, and V = the number of vertices. Therefore a 16-faced polyhedron has 24 edges.

14. Lesson Fourteen

Time	Topic
	Problem Solving (Mixed Problems)
10 min.	Problem Five From Lesson Thirteen

If there had been insufficient time to complete problem five the previous day, it will be taken up first in this lesson. The solution to this problem will be given by the teacher using the overhead projector and prepared transparencies; it is given in Lesson Thirteen under the heading: "Answers to Problems From the Thirteenth Day".

30 min.	Instructional Strategy For Problems Six and Seven

The instructor will allow the students about twenty minutes to solve these problems. He will then show them the complete solutions on parepared transparencies. (The complete solutions to problems six and seven are presented in this lesson under the heading: "Answers to Problems From the Fourteenth Day").

FOURTEENTH DAY

The two problems that were discussed in this lesson are all presented in Lesson Twelve under the heading: "Twelfth Day".

ANSWERS TO PROBLEMS FROM THE FOURTEENTH DAY

Problem 6

Unknown?

- the number of hens and rabbits

Given?

- farmer has hens and rabbits
- they have 50 heads and 140 feet.

Related Problem?

- characteristics - two comparisons can be made between the hens and the chickens.
  - the problem can be easily translated into mathematics sentences.

- method of solution - Decomposing and Recombining

- Restated problem?

  Let x be the number of hens.

  Let y be the number of rabbits.

ORIGINAL PROBLEM	ALGEBRAIC TRANSLATION
A farmer has hens and rabbits.	x, y
These animals have 50 heads	$x+y=50$
and 140 feet.	$2x+4y=140$
How many hens and how many rabbits has the farmer?	x, y ?

$$2x+4y = 140 \quad \text{(call this equation \#1)}$$

$$x+y = 50 \quad \text{(call this equation \#2)}$$

Divide equation #1 by 2.

$$x+2y = 70 \quad \text{(call this equation \#3)}$$

Subtract equation #2 from equation #3.

$$y = 20$$

Solve for x in equation #1.

$$2x+4y = 140$$

$$2x+4(20) = 140$$

$$2x = 60$$

$$x = 30$$

Check both solutions in equation #2.

$$x+y = 50$$

$$30+20 = 50$$

$$50 = 50$$

Therefore there are 30 hens and 20 rabbits.

Problem 7

Unknown?

- truth of theorem

Given?

- r is radius of inscribed circle
- R is radius of circumscribed circle
- H is the longest altitude of given triangle

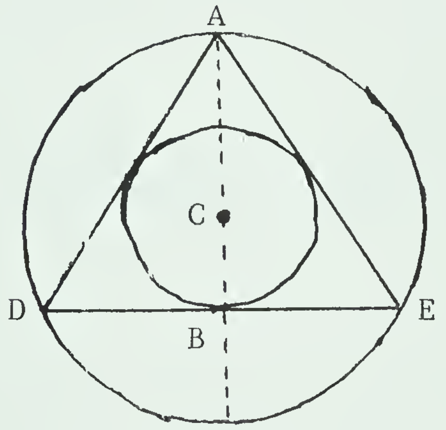
Related Problem?

- characteristics - choice to be made between two alternatives (theorem is true or false)
  - specialized problem will either give a clue as to the choice or decide the choice

- method of solution - Specialization

- Specialized problem?

  Given an equilateral triangle with an inscribed circle (with radius r), a circumscribed circle (with radius R), and its longest altitude (H), is $r+R \lessgtr H$?



$r = H/3$
$R = (2H)/3$

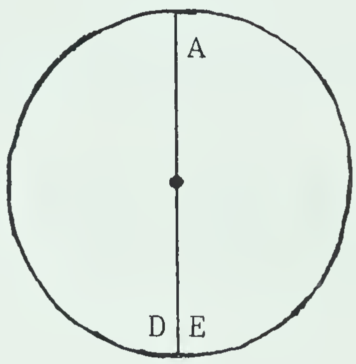
$\overline{AB}$ = median of equilateral triangle
$= H$
Therefore $2(\overline{BC}) = \overline{AC}$

Thus $r = \overline{BC}$
$= H/3$
$R = \overline{AC}$
$= (2H)/3$

In this case, the theorem is true.

A more extended case is that of the isosceles

triangle. The form of an isosceles triangle varies with the angle at the vertex and there are two extreme (or limiting) cases; the one in which the angle at the vertex becomes 0 degrees, and the other in which it becomes 180 degrees. In the first extreme case the base of the isosceles triangle vanishes and



$r = 0$

$R = H/2$

For this case, the theorem is true.

In the second limiting case, however, the height vanished and

$r = 0$

$R = \infty$ (Circumscribed circle must pass through points A, D, and E.

$H = 0$

For this case, the theorem is not true. Therefore the theorem is false, and the problem is solved.

15. Lesson Fifteen

(a) Lesson Plan

Time	Topic
	Characteristics and Solution-Methods Quiz
15 min.	Problem Seven From Lesson Fourteen If there had been insufficient time to complete problem seven the previous day, it will be taken up first in this lesson. The solution to this problem will be given by the teacher using the overhead projector and prepared transparencies (or using the blackboard); it is given in Lesson Fourteen under the heading: "Answers to Problems From the Twelfth Day".
25 min.	Administration of the Quiz The instructor will hand out the quiz and in doing so, he will tell the students to fill in their names on the first page of the quiz.

CHARACTERISTICS AND SOLUTION-METHODS QUIZ

Time: 20 minutes

Instructions: This is a multiple-choice problem-solving quiz. Its purpose is to examine the students' ability to choose the key characteristics of each problem, and then choose the correct method of solution to follow in solving each problem.
STUDENTS ARE NOT ASKED TO WORK OUT AND WRITE DOWN A COMPLETE SOLUTION TO EACH PROBLEM but rather to choose a METHOD OF SOLUTION that they feel will lead to the correct solution to the problem.

Problem 1

In this two-person game, the players alternately place poker chips on a circular table. The chips must not overlap and must be completely on the table; that is, no poker chip may stick out over the edge of the table. The last player to play a chip on the table is the winner. If each player makes the optimal move on his turn,

will the first player or the second player be the winner?

(a) Write down the characteristics of this problem.

(b) Underline the correct method of solution for this problem. Generalization Specialization Analogy Decomposing and Recombining Working Backward

Problem 2

Fifteen pennies are placed on a table in front of two players. Each player is allowed to remove at least one penny but not more than five pennies at his turn. The players alternate turns, each removing from one to five pennies n number of turns, until one player takes the last penny on the table, and wins all 15 pennies. Is there a method of play that will guarantee victory? If so, what is it?

(a) Write down the characteristics of this problem.

(b) Underline the correct method of solution for this problem. Generalization Specialization Analogy Decomposing and Recombining Working Backward

Problem 3

Find the altitude of a regular tetrahedron with given edge 10 cm. (Hint: the lower end of the altitude intersects a median of the triangular base into two line segments whose lengths are in the ratio 2:1).

(a) Write down the characteristics of this problem.

(b) Underline the correct method of solution for this problem. Generalization Specialization Analogy Decomposing and Recombining Working Backward

Problem 4

How many kilograms of candy worth $1.50 per kilogram and how many kilograms of candy worth $2.70 per kilogram must be used to make a 225 kilogram mixture which is worth $2.28 per kilogram?

(a) Write down the characteristics of this problem.

(b) Underline the correct method of solution for this problem. Generalization Specialization Analogy Decomposing and Recombining Working Backward

Problem 5

Find the number of lines determined by 14 points, no 3 of which are collinear.

(a) Write down the characteristics of this problem.

(b) Underline the correct method of solution for this problem. Generalization Specialization Analogy Decomposing and Recombining Working Backward

Problem 6

Three people play a game in which one person loses and two people win each game. The one who loses must double the amount of money that each of the other two players has at that time. The three players agree to play three games. At the end of the three games, each player has lost one game and each person has $8. What was the original stake of each player?

(a) Write down the characteristics of this problem.

(b) Underline the correct method of solution for this problem. Generalization Specialization Analogy Decomposing and Recombining Working Backward

Problem 7

Find a way to determine the sum of the odd numbers less than 100.

(a) Write down the characteristics of this problem.

(b) Underline the correct method of solution for this problem. Generalization Specialization Analogy Decomposing and Recombining Working Backward

Problem 8

In a triangle, let r be the radius of the inscribed circle, R the radius of the circumscribed circle, and H the longest altitude. Then $R+r \leq H$. Prove or disprove this theorem.

(a) Write down the characteristics of this problem.

(b) Underline the correct method of solution for this problem. Generalization Specialization Analogy Decomposing and Recombining Working Backward

Problem 9

Anne is 7 years older than Jane. One year ago, she was twice as old as Jane. How old is each now?

(a) Write down the characteristics of this problem.

(b) Underline the correct method of solution for this problem. Generalization Specialization Analogy Decomposing and Recombining Working Backward

Problem 10

John has a cylindrical can in which he wants to keep his pencils. He knows that the volume of the can is 332.75 $cm.^3$ and its height is 3.5 cm. What is the longest pencil (theoretically speaking) that John could put in the can, and still be able to place the lid firmly onto the top? (π is given as 22/7)

(a) Write down the characteristics of this problem.

(b) Underline the correct method of solution for this problem. Generalization Specialization Analogy Decomposing and Recombining Working Backward

(b) Scoring Procedure For the Characteristics and Solution-Methods Quiz

Problem 1 (2 marks)

Characteristics (½ mark each)
- A decision needs to be made between two alternatives (Will the first or second player consistently win the game?).
- A specialized problem could give a clue as to the choice (to be made between the two alternatives) or decide the choice.

Method of Solution - Specialization (1 mark)

Problem 2 (2 marks)

Characteristics (½ mark each)
- The outcome is known (the winning player faces one to five pennies on the table, and it is his turn).
- The problem has a trial-and-error beginning.

Method of Solution - Working Backward (1 mark)

Problem 3 (2 marks)

Characteristics (½ mark each)
- The problem involves a three-dimensional figure (a rectangular solid), but is asking for a one-dimensional answer (diagonal).
- Using a triangle (two-dimensional figure) simplifies the problem.

Method of Solution - Analogy (1 mark)

Problem 4 (2 marks)

Characteristics (½ mark each)
- A problem in which two comparisons are made between two kinds of candy;

thus two equations can be established in order to generate the solution.
- A problem which is easily translated into two mathematics sentences.

Method of Solution - Decomposing and Recombining
(1 mark)

Problem 5 (2 marks)

Characteristics ($\frac{1}{2}$ mark each)
- The number of points is large; the estimated answer is very large.
- To draw a diagram depicting the situation, and to attempt to count all the resulting lines is an extremely difficult task.

Method of Solution - Generalization
(1 mark)

Problem 6 (2 marks)

Characteristics ($\frac{1}{2}$ mark each)
- The outcome is known (each of the three players has \$8).
- The problem has a trial-and-error beginning.

Method of Solution - Working Backward
(1 mark)

Problem 7 (2 marks)

Characteristics ($\frac{1}{2}$ mark each)
- The number of odd numbers is large; the estimated answer is very large.
- To add all the odd numbers together is an extremely difficult and laborious task.

Method of Solution - Generalization
(1 mark)

Problem 8 (2 marks)

Characteristics - Choice to be made between two
(½ mark each) alternatives (theorem is true or false).
- Specialized problem will either give a clue as to the choice or decide the choice.

Method of Solution - Specialization
(1 mark)

Problem 9 (2 marks)

Characteristics - A problem in which two comparisons
(½ mark each) are made between Anne's and Jane's ages; thus two equations can be established in order to generate the solution.
- A problem which is easily translated into two mathematics sentences.

Method of Solution - Decomposing and Recombining
(1 mark)

Problem 10 (2 marks)

Characteristics - Given a three-dimensional figure
(½ mark each) (cylindrical can); wanted a one-dimensional answer (length of pencil).
- The three-dimensional figure can be easily simplified into a two-dimensional figure (rectangle).

Method of Solution - Analogy
(1 mark)

16. Lesson Sixteen

Time	Topics
	1. Review of Characteristics and Solution-Methods Quiz.
	2. Questioning Time.

10 min.

Before handing back the quiz, the instructor will mention two points that emerged from piloting the Characteristics and Solution-Methods Quiz. First, when the students write down the characteristics of the problem before them, they must state them more explicitly. For example, instead of characterizing problem two (from the quiz) as: Outcome is known and trial-and-error beginning, the problem solver should describe the problem thus: Outcome is known (the winning player faces one to five pennies on the table, and it is his turn) and trial-and-error beginning.

A second point that arose from piloting the quiz is that there is some confusion between the methods of solution, specialization and working backward. The problem characteristics that are typically associated with specialization are:

1. A decision needs to be made between two alternatives.

2. A specialized problem could give a clue as to the choice (to be made between the two alternatives) or decide the choice.

The problem characteristics that are typically associated with the method of solution, working backward, are:

1. The outcome is known.

2. The problem has a trial-and-error beginning.

For instance, in problem two (from the quiz) there are two questions that are asked: "Is there a method of play that will guarantee victory?" and "If so, what is it?". Since the first question requires a yes or no response, some students concluded that specialization was the method of solution to be used in solving that problem. The second question in problem two requests a procedural answer. That is, a method of play that will guarantee victory must be found. The problem solver initially needs to search for this procedure, and then subsequently he will be able to answer: "Yes there is a method." or "No there is no method that will guarantee victory." Therefore, working backward is the correct solution method for solving problem two.

15 min. After discussing the above two points, the teacher will give back the answer sheets to the Characteristics and Solution-Methods Quiz. He will use Lesson Fifteen, subsection (b) entitled "Scoring Procedures For the

Characteristics and Solution-Methods Quiz" in his discussion of the quiz.

15 min. For the remaining fifteen minutes in the period, the teacher will allow the students to ask him questions concerning any part of the problem-solving unit that they have just completed.

17. Lesson Seventeen

(a) Lesson Plan

Time	Topic
	Heuristic Problem-Solving Test
80 min.	Administration of the Test
	The teacher will hand out the tests and in doing so, he will tell the students to fill in their name on the first page of the test. (The Heuristic Problem-Solving Test is contained in this thesis in Appendix C).

(b) Scoring Procedure For the Heuristic Problem-Solving Test

Problem 1 (6 marks)

Let x km./hr. be the rate of one airplane.

Let y km./hr. be the rate of the other airplane.

(1 mark for naming both unknowns. If the units are missing after both x and y, then minus ½ mark).

$$4x + 4y = 8000 \text{ or}$$

$$x+y = 2000 \quad \text{(Call this equation \#1)}$$

$$y = 3x \quad \text{(Call this equation \#2)}$$

(2 marks; 1 mark for each equation)

Substitute equation #2 into equation #1.

$$x+(3x) = 2000 \quad \text{(1 mark for correct simplification)}$$

$$x = 500 \text{ km./hr.} \quad \text{(1 mark; minus ½ mark if units are missing)}$$

Solve for y in equation #1.

$$(500)+y = 2000$$

$$y = 1500 \text{ km./hr.} \quad \text{(1 mark; minus ½ mark if units are missing)}$$

Check in equation #2.

$$1500 = 3(500)$$

Therefore the rates of the two planes were 500 km./hr. and 1500 km./hr.

<u>Problem 2</u> (6 marks)

To prove: $1/(2!) + 2/(3!) + \ldots + n/((n+1)!)$
$= 1 - n/((2n-1)!)$

Proof: Take a specialized case, for example $n=2$, to work through. This will both help the problem-solver's understanding of the equation and also perhaps, disprove the equation.

(2 marks for choosing a specialized case that will disprove the equation)

$$1/(1+1)! + 2/(2+1)! = 1 - 2/(4-1)!$$

$$1/2! + 2/3! = 1 - 2/3!$$

$$1/2 + 2/6 = 1 - 2/6$$

$$5/6 \neq 4/6$$

(1 mark for demonstrating a correct understanding of the given equation)

(1 mark for correcting substituting for n into the given equation)

(1 mark for correctly simplifying both the left side and the right side of the equation)

Therefore the equation is incorrect and thus it is disproven.

(1 mark for making the concluding statement)

Problem 3 (6 marks)

	MAN ENTERS STORE	MAN RECEIVES MONEY	MAN SPENDS MONEY	MAN LEAVES STORE
FIRST STORE	\$8.75	\$8.75	\$10.00	\$7.50
SECOND STORE	\$7.50	\$7.50	\$10.00	\$5.00
THIRD STORE	\$5.00	\$5.00	\$10.00	\$0.00

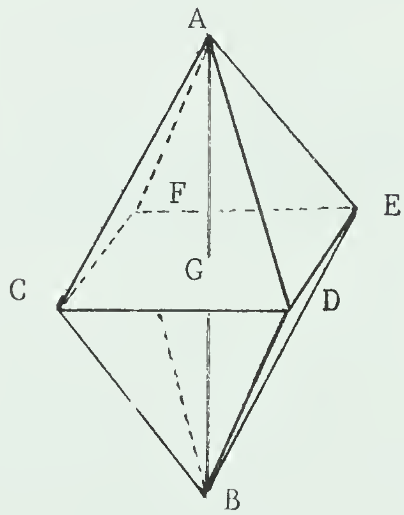
Therefore the man started with \$8.75.

(2 marks for the third row in the chart; that is, $\frac{1}{2}$ mark for each column #1 - #4)

(2 marks for the second row in the chart; that is, $\frac{1}{2}$ mark for each column #1 - #4)

(2 marks for the first row in the chart; that is, $\frac{1}{2}$ mark for each column #1 - #4)

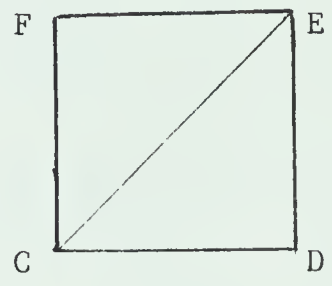
Problem 4 (6 marks)



Problem 4 (First Solution)

Diagonal = $\overline{AB}$ = $\overline{CE}$ = $\overline{FD}$

Analogous Problem: Given the square CDEF, find $\overline{CE}$ or $\overline{FD}$.



(2 marks for square CDEF)

$$\overline{CE}^2 = \overline{CD}^2 + \overline{DE}^2 \quad \text{(1 mark)}$$

$$= 10^2 + 10^2 \quad \text{(1 mark)}$$

$$\overline{CE} = \sqrt{200}$$

$$= 10\sqrt{2} \text{ cm.}$$

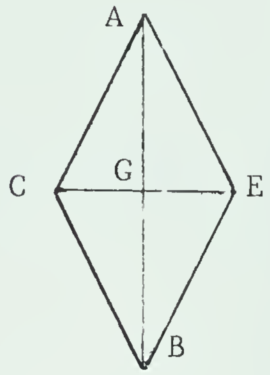
$$= 14.14 \text{ cm.}$$

(2 marks for answer)

Therefore, the length of the diagonal of the regular octahedron is $10\sqrt{2}$ cm. or 14.14 cm.

Problem 4 (Second Solution)

Take the analogous problem: Given a rhombus AEBC, find the length of the vertical diagonal $\overline{AB}$.



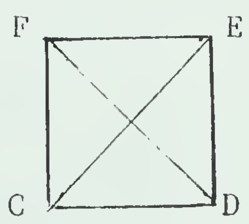
Given $\overline{AC} = \overline{CB} = \overline{BE} = \overline{EA}$.
Find $\overline{AB}$.

First: Triangle AGC is a right triangle (because AEBC is a rhombus). Therefore:

$$(\overline{AC})^2 = (\overline{CG})^2 + (\overline{AG})^2 \qquad \text{(Pythagoras)}$$
$$= (\overline{CE}/2)^2 + (\overline{AB}/2)^2$$
$$= (\overline{CE})^2/4 + (\overline{AB})^2/4$$
$$(\overline{AB})^2 = 4(\overline{AC})^2 - (\overline{CE})^2$$
$$\overline{AB} = \sqrt{4(\overline{AC})^2 - (\overline{CE})^2}$$
$$= \sqrt{4(10)^2 - (\overline{CE})^2}$$
$$= \sqrt{400 - (\overline{CE})^2} \qquad \text{(2 marks)}$$

Second: Find $\overline{CE}$.

Examine square CDEF in the diagram of the regular octahedron.



$$\overline{CE} = \sqrt{(\overline{DE})^2 + (\overline{CD})^2} \qquad \text{(Pythagoras)}$$
$$= \sqrt{(10)^2 + (10)^2}$$
$$= \sqrt{200}$$
$$= 10\sqrt{2} \text{ cm.} \qquad \text{(2 marks)}$$

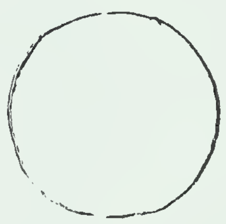
Third: but $\overline{AB} = \sqrt{400 - (\overline{CE})^2}$

$$= \sqrt{400 - (10\sqrt{2})^2}$$
$$= \sqrt{400 - 100(2)}$$
$$= \sqrt{200}$$
$$= 10\sqrt{2} \text{ cm. or } 14.14 \text{ cm.} \qquad \text{(2 marks)}$$

Therefore the length of the diagonal of the regular octahedron is $10\sqrt{2}$ cm. or 14.14 cm.

Problem 5 (6 marks)

Case #1 -



Given 0 chords, there is 1 region formed.

Case #2 -

Given 1 chord, there are 2 regions formed.

Case #3 -

Given 2 chords, there are 4 regions formed.

Case #4 -  Given 3 chords, there are 7 regions formed.

(2 marks for the related problems; ½ mark for each)

NUMBER OF CHORDS	NUMBER OF REGIONS
0	1
1	2
2	4
3	7
4	11
5	16
6	22
7	29
8	37
9	46
10	56
11	67
12	79
13	92
14	106
15	121
16	137
17	154
18	172
19	191
20	211

(2 marks for recognizing the pattern; ½ mark for each of the answers in the 5th through 8th rows)

(2 marks for the correct answer)

Therefore, given 20 chords, there are 211 regions formed (maximum).

18. Lesson Eighteen

(a) Lesson Plan

Time	Topic
	Characteristics and Solution-Methods Test
20 min.	Administration of the Test
	The teacher will hand out the tests and in doing so, he will tell the students to fill in their name on the first page of the test. (The Characteristics and Solution-Methods Test is contained in this thesis in Appendix D).

(b) Scoring Procedure For the Characteristics and Solution-Methods Test

Problem 1 (2 marks)

Characteristics ($\frac{1}{2}$ mark each) - A problem in which two comparisons are made between two airplanes.
- A problem which is easily translated into two mathematics sentences.

Method of Solution (1 mark) - Decomposing and Recombining

Problem 2 (2 marks)

Characteristics ($\frac{1}{2}$ mark each) - A decision needs to be made between two alternatives (that is, to prove or to disprove the equation).
- A specialized problem could give a clue or decide the choice to be made between the two alternatives.

Method of Solution (1 mark) - Specialization

Problem 3 (2 marks)

Characteristics ($\frac{1}{2}$ mark each) - The outcome is known (the man has no money left).
- The problem has a trial-and-error beginning.

Method of Solution (1 mark) - Working Backward

Problem 4 (2 marks)

Characteristics (½ mark each)
- The problem involves a three-dimensional figure (a regular octahedron), but is asking for a one-dimensional answer (diagonal).
- Using a rhombus (a two-dimensional figure) simplifies the problem.

Method of Solution (1 mark) - Analogy

Problem 5 (2 marks)

Characteristics (½ mark each)
- The number of chords is large; the resulting number of regions is very large.
- To draw a diagram depicting the situation, and to attempt to count all the resulting regions is an extremely difficult task.

Method of Solution (1 mark) - Generalization

Problem 6 (2 marks)

Characteristics (½ mark each)
- Given a three-dimensional figure (cylindrical can); wanted a one-dimensional answer (length of pencil).
- the three-dimensional figure can be easily simplidied into a two-dimensional figure (rectangle).

Method of Solution (1 mark) - Analogy

Problem 7 (2 marks)

Characteristics (½ mark each)
- A problem in which two comparisons are made between the two unknowns (two sides of an isosceles triangle).

- A problem which is easily translated into two mathematics sentences.

Method of Solution - Decomposing and Recombining
(1 mark)

Problem 8 (2 marks)

Characteristics (½ mark each)
- The number of discs is large; the resulting number of moves is very large.
- To draw a diagram depicting the situation, and to attempt to count all the moves is an extremely difficult task.

Method of Solution - Generalization
(1 mark)

Problem 9 (2 marks)

Characteristics (½ mark each)
- A decision needs to be made between two alternatives (that is, to prove or to disprove the statement in the problem.
- A specialized problem could give a clue or decide the choice to be made between the two alternatives.

Method of Solution - Specialization
(1 mark)

Problem 10 (2 marks)

Characteristics (½ mark each)
- The outcome is known (opponents turn and there is only one toothpick left).
- The problem has a trial-and-error beginning.

Method of Solution - Working Backward
(1 mark)

19. Lesson Nineteen

Time	Topic
	Mathematics 20 Problem-Solving Test (post-test)
70 min.	Administration of the Test

The teacher will hand out the tests and in doing so, he will tell the students to fill in their name on the first page of the test. The Mathematics 20 Problem-Solving Test is contained in this thesis in Appendix A and the Scoring Procedure For the Mathematics 20 Problem-Solving Achievement Test is contained in this thesis in Appendix G (lesson one).

APPENDIX H

LESSON PLANS FOR THE CONVENTIONAL APPROACH
TO PROBLEM SOLVING

1. Lesson One

Time	Topic
	Mathematics 20 Problem-Solving Test (pretest)
70 min.	Administration of the Test

The instructor will hand out the tests and in doing so, he will tell the students to fill in their names on the first page of the test. The test is contained in this thesis in Appendix A, and the scoring procedures are contained in Appendix G (lesson one).

2. Lesson Two

Time	Topic
	Problem Solving (type #1)
40 min.	Lesson Plan Outline

The teacher will give out the booklets of twenty-four problems, one to each member of the class. He will then do problem one for the students in order to demonstrate the solution strategy for type #1 problems. Subsequently, he will ask the students to work on problem two during the period and to finish it for homework.

BOOKLET OF PROBLEMS

Problem 1

In a triangle ABC, if 20 lines are drawn from vertex A through points on the opposite side, how many triangles are formed?

Problem 2

Find the number of lines determined by 14 points, no 3 of which are collinear.

Problem 3

How many angles are formed when 15 rays are drawn from the same end point?

Problem 4

Find a way to determine the sum of the odd numbers less than 100.

Problem 5

```
            E
          E S
        E S K
      E S K I
    E S K I M
  E S K I M O
E S K I M O S
```

Given the grid, as shown on the left, how many different ways can the word ESKIMOS be made? (The word is made by moving in a vertical and/or horizontal direction on the grid).

Problem 6

Find the sum of the first one hundred natural numbers.

Problem 7

Find the number of edges of a regular 16-faced polyhedron. (Hint: this polyhedron has 10 vertices).

Problem 8

Given a jar that will hold exactly 9 quarts of water, a jar that will hold exactly 4 quarts of water, no other containers holding water, but an infinite supply of water, describe a sequence of fillings and emptyings of water jars that will result in achieving 6 quarts of water.

Problem 9

Fifteen pennies are placed on a table in front of two players. Each player is allowed to remove at least one penny but not more than five pennies at his turn. The players alternate turns, each removing from one to five pennies n number of turns, until one player takes the last penny on the table, and wins all 15 pennies. Is there a method of play that will guarantee victory? If so, what is it?

Problem 10

Three people play a game in which one person loses and two people win each game. The one who loses must double the amount of money that each of the other two players has at that time. The three players agree to play three games. At the end of the three games,

each player has lost one game and each person has $8. What was the original stake of each player?

Problem 11

On an infinitely extended checkerboard, one is given three black checkers and two white checkers initially placed in immediately adjacent squares on a single row, proceeding from left to right, as shown in the figure on the next page: black (B), white (W), black (B), white, black. The problem is to transform this arrangement of alternating black and white checkers into an arrangement in which all three black checkers are on the left and both white checkers are on the right (BBBWW), with all checkers being in adjacent squares and in the same row (see the figure below). The allowable operation is to move two adjacent checkers at a time, one of which must be a black checker and one a white checker. During a move, the two checkers being moved must remain together at all times, with no reversal of their left-to-right order. You are permitted to move a white-black or black-white pair of checkers to any adjacent pair of unoccupied squares along the same line. Note that there is no need to keep the checkers that are not being moved in immediately adjacent squares at any time. That is, there may be unoccupied squares between checkers at various stages between the givens and the goal. Also note that the five checkers in the goal state need not occupy the same five squares on the checkerboard as they did in the given state. They may occupy any immediately

adjacent five square in the same row.

GIVEN STATE GOAL STATE

Problem 12

Bob has 10 pockets and 44 silver dollars. He wants to put his dollars into his pockets so distributed that each pocket contains a different number of dollars. Can he do so?

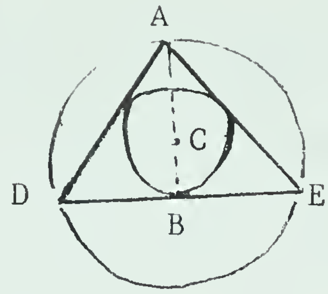
Problem 13

In this two-person game, the players alternately place a poker chip on a circular table. The chips must not overlap and must be completely on the table; that is, no poker chip may stick out over the edge of the table. The last player to play a chip on the table is the winner. If each player makes the optimal move on his turn, will the first player or the second player be the winner?

Problem 14

In a triangle, let r be the radius of the inscribed circle, R the radius of the circumscribed circle, and H the longest altitude. Then $r+R \leq H$. Prove or disprove this theorem.

Hints: 1.



Triangle ADE is any type of triangle.

2. $\overline{AB}$ is the altitude of the triangle ADE. If $\overline{AB}$ is also the median of triangle ADE (that is, triangle ADE is an equilateral triangle), then point C intersects the median of the triangular base into two line segments whose lengths are in the ratio 2:1.

Problem 15

Find the altitude of a regular tetrahedron with given edge 10 cm. (Hint: the lower end of the altitude intersects a median of the trianglular base into two line segments whose lengths are in the ratio 2:1).

Problem 16

Find the diagonal of a rectangular solid of which the length, width, and height are "a" cm , "b" cm , and "c" cm respectively.

Problem 17

Bill and Doug are preparing for an international math competition, to be administered the following week. They are asking each other questions. One problem that Bill gave Doug to solve was real challenging. It stated: "A boy asked his father how much money he (the

father) had in his wallet. The father's reply was in the form of a riddle. He said: the number of dollars that I have is multiplied by itself four times and the result multiplied by four. From this amount, thirty-three times the square of the original amount is subtracted. The result is $27. How many dollars did he have in his wallet?"

Problem 18

A man spends 18% of his monthly income for rent. If his rent is $45 a month, what is his salary for a month?

Problem 19

Anne is 7 years older than Jane. One year ago, she was twice as old as Jane. How old is each now?

Problem 20

The difference between two numbers is 10. If twice the larger is subtracted from five times the smaller, the remainder is 22. What are the numbers?

Problem 21

How many kilograms of candy worth $1.50 per kilogram and how many kilograms of candy worth $2.70 per kilogram must be used to make a 225 kilogram misture which is worth $2.28 per kilogram?

Problem 22

John has a dream about airplane pilots and sports cars. When all the airplane pilots in John's dream got into sports cars, there was one pilot per car, and seven cars were still empty. Then one half of the pilots departed by jet for Hawaii. After that, when all of the remaining pilots got into sports cars, there was one pilot per car, but twenty-nine cars were empty. How many sports cars were there in John's dream?

Problem 23

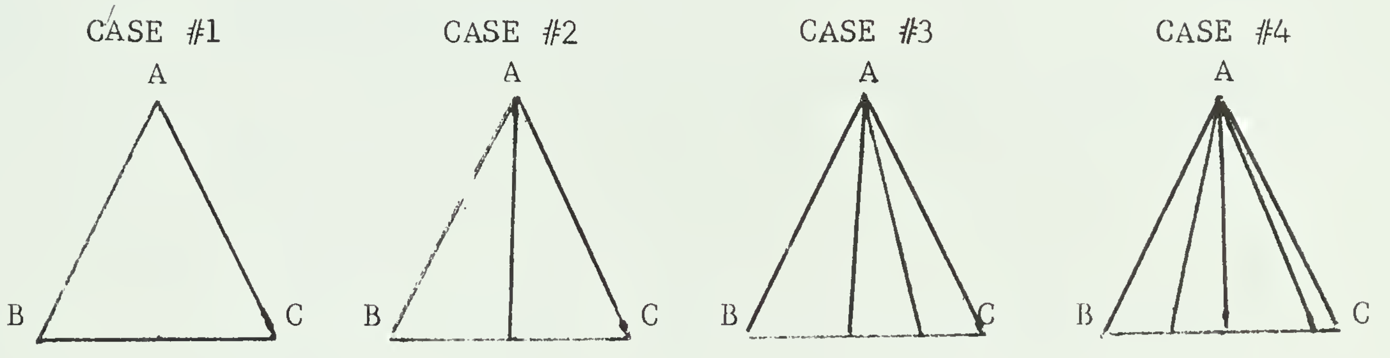
A lady has one sum of money invested at 5% per annum and a second sum, $1500 larger than the first, invested at 6% per annum. Her total income from these sums is $200 (after one year of investment). How much has she invested at each rate?

Problem 24

A farmer has hens and rabbits. These animals have 50 heads and 140 feet. How many hens and how many rabbits has the farmer?

ANSWER TO PROBLEM FROM THE SECOND DAY

Problem 1



Number of Lines	Number of Triangles
0	1
1	3
2	6
3	10
4	15
5	21
6	28
7	36
8	45
9	55
10	66
11	78
12	91
13	105
14	120
15	136
16	153
17	171
18	190
19	210
20	231

Therefore, if 20 lines are drawn, there are 231 triangles formed.

3. Lesson Three

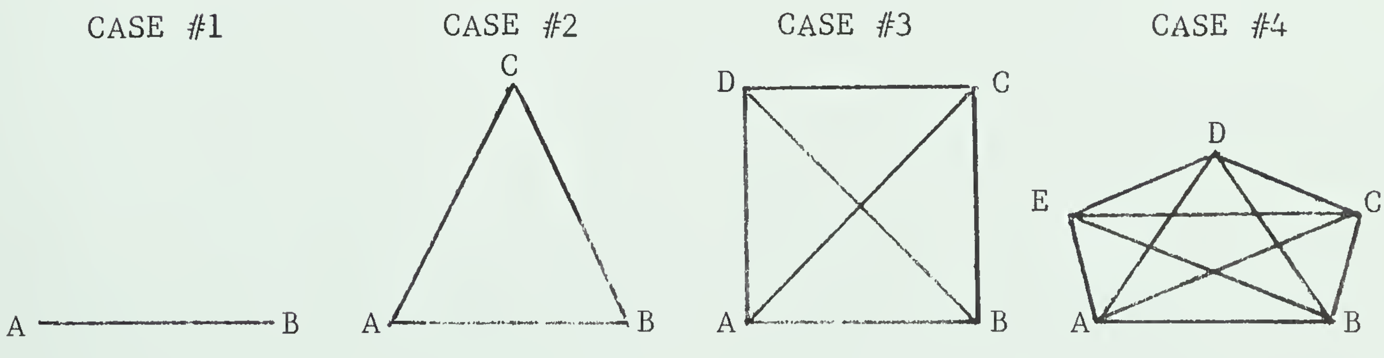
Time	Topic
	Problem Solving (type #1)
40 min.	Lesson Plan Outline

The teacher will correct problem two (with input from the students), writing out the solution on the blackboard. The solution is located in this lesson under the heading: "Answer to Problem For the Third Day".

He will then assign problems three and four to be worked on during the period and to be completed for the next lesson.

ANSWER TO PROBLEM FOR THE THIRD DAY

Problem 2



Number of Points	Number of Lines
2	1
3	3
4	6
5	10
6	15
7	21
8	28
9	36
10	45
11	55
12	66
13	78
14	91

Therefore, given 14 points, there are 91 lines that can be drawn.

4. Lesson Four

Time — Topic

Problem Solving (type #1)

40 min. — Lesson Plan Outline

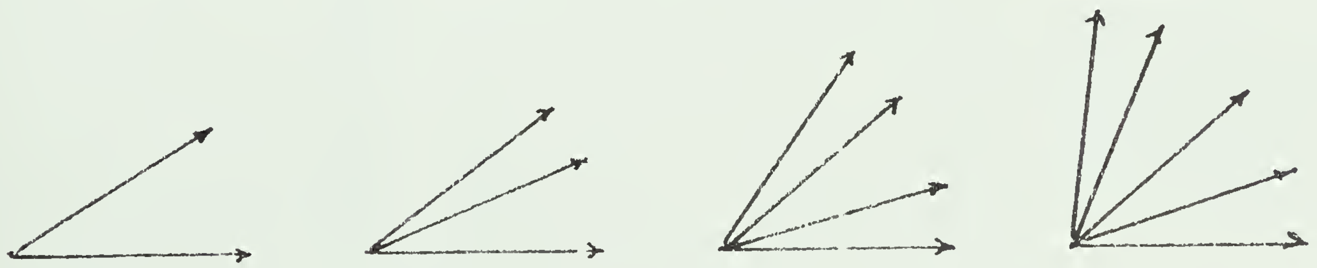
The teacher will correct problems three and four (with input from the students), writing out the solutions on the blackboard. The solutions are located in this lesson under the heading: "Answers to Problems For the Fourth Day".

He will then assign problems five and six to be worked on during the period and to be completed for the next lesson.

ANSWERS TO PROBLEMS FOR THE FOURTH DAY

Problem 3

CASE #1 CASE #2 CASE #3 CASE #4



Number of Rays	Number of Angles Formed
2	2
3	6
4	12
5	20
6	30
7	42
8	56
9	72
10	90
11	110
12	132
13	156
14	182
15	210

Therefore, there are 210 angles formed.

Problem 4

Case #1 - $1 = 1$

Case #2 - $1+3 = 4 = 2^2$

Case #3 - $1+3+5 = 9 = 3^2$

Case #4 - $1+3+5+7 = 16 = 4^2$

Number of Odd Numbers Added	Sum of the Odd Numbers
1	1
2	$4=2^2$
3	9
4	$16=4^2$
.	
.	
.	
50	50^2

Therefore, given the first 50 odd numbers, their sum is 50^2 or 2500.

5. Lesson Five

Time	Topic
	Problem Solving (type #1)

40 min.

Lesson Plan Outline

The teacher will correct problems five and six (with input from the students), writing out the solutions on the blackboard. The solutions are located in this lesson under the heading: "Answers to Problems For the Fifth Day".

He will then assign problem seven to be worked on during the period and to be completed for the next lesson.

ANSWERS TO PROBLEMS FOR THE FIFTH DAY

Problem 5

```
Case #1 -  E

             E
Case #2 -  E S

               E
             E S
Case #3 -  E S K

                 E
               E S
             E S K
Case #4 -  E S K I
```

Number of Letters Used	Number of Words Made
1	1
2	2
3	4
4	8
5	16
6	32
7	64

Therefore, given 7 letters on the grid, there are 64 words that can be made.

Problem 6

Case #1 - 1 = 1

Case #2 - 1+2 = 3 = (1.5)(2)

Case #3 - 1+2+3 = 6 = (2)(3)

Case #4 - 1+2+3+4 = 10 = (2.5)(4)

Number of Natural Nos.	Sum of the Natural Nos.
1	1
2	(1.5)(2)=3
3	(2)(3)=6
4	(2.5)(4)=10
.	
.	
.	
100	(50.5)(100)=5050

Therefore, given the first 100 natural numbers, their sum is 5050.

6. Lesson Six

Time — Topic

Problem Solving (types #1 and #2)

40 min. — Lesson Plan Outline

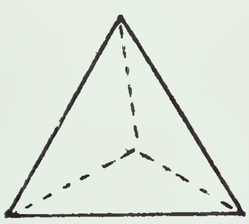
The teacher will correct problem seven (with the input from the students), writing out the solution on the blackboard. He will then do problem eight for the pupils in order to demonstrate the solution strategy for type #2 problems. The solutions for problems seven and eight are located in this lesson under the heading: "Answers to Problems For the Sixth Day".

The instructor will then assign problems nine and ten to be worked on during the period and to be completed for the next lesson.

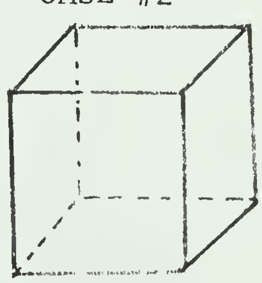
ANSWERS TO PROBLEMS FOR THE SIXTH DAY

Problem 7

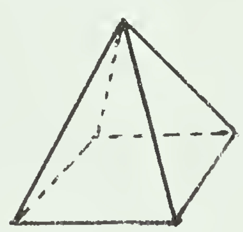
CASE #1



CASE #2



CASE #3



Case #1 is a tetrahedron, having 4 faces, 4 vertices, and 6 edges.

Case #2 is a cube, having 6 faces, 8 vertices, and 12 edges.

Case #3 is a pyramid, having 5 faces, 5 vertices, and 8 edges.

NAME OF SOLID	NUMBER FACES	NUMBER VERTICES	NUMBER EDGES
Tetrahedron	4	4	6
Cube	6	8	12
Pyramid	5	5	8
.			
.			
.			
16-Faced Polyhedron	16	10	24

By examining the corresponding numbers in columns two and three in the above table, a pattern can be found. This pattern can be summarized by the formula E=F+V-2, where E = the number of edges, F = the number of faces, and V = the number of vertices. Therefore, a 16-faced polyhedron has 24 edges.

Problem 8

1. Question: "If you have six quarts of water in the nine-quart pail, where did it come from?"

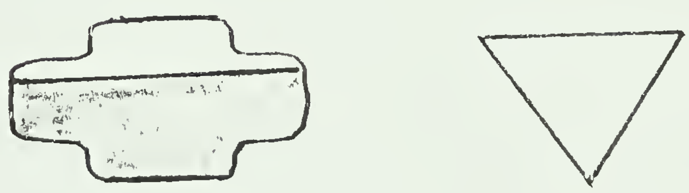
   Reply: "The water came from the nine quarts of water in the full pail of which three quarts were removed".

2. Question: "How were the three quarts removed?"

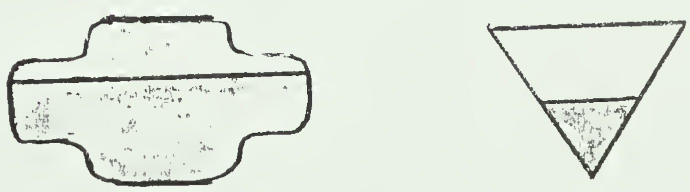
   Reply: "The water (three quarts) was poured out of the full nine-quart pail into the four-quart pail in which there was already one quart".

3. Question: "Where did the one quart of water, in the four-quart pail, come from?"

   Reply: "The one quart of water came as a result of pouring out eight quarts of water (filling the four-quart pail twice) from the full nine-quart pail. Then pouring the remaining one quart into the empty four-quart pail".



First Diagram



Second Diagram

Third Diagram

Conclusion: Take a nine-quart pail, full of water, and pour out eight quarts (filling the four-quart pail twice). Then pour the remaining one quart into the empty four-quart pail. Next, refill the nine-quart pail. Pour out the water from the nine-quart pail so as to fill the four-quart pail. There are now only six quarts remaining in the nine-quart pail.

7. Lesson Seven

Time	Topic
	Problem Solving (type #2)
40 min.	Lesson Plan Outline

The teacher will correct problems nine and ten (with input from the students), writing out the solutions on the blackboard. The solutions are located in this lesson under the heading: "Answers to Problems For the Seventh Day".

He will then assign problem eleven to be worked on during the period and to be completed for the next lesson.

ANSWERS TO PROBLEMS FOR THE SEVENTH DAY

Problem 9

1. Question: "If the ultimate winner wants to face one to five pennies, how many pennies should he leave the ultimate loser confronted with?"

   Reply: "If the ultimate loser was confronted with six pennies, then no matter how many pennies he took (from one to five), there would still be from one to five pennies left on the table, giving the ultimate winner, his turn and thus the victory".

2. Question: "If the ultimate winner wants to face seven to eleven pennies, how many pennies should he leave the opposing player confronted with?"

   Reply: "If the ultimate loser was confronted with twelve pennies, after the ultimate winner's preceding move, then no matter how many pennies the opposing player took (from one to five), the ultimate winner would be able to take enough pennies to confront him with six pennies on his next turn".

3. Question: "What is the ultimate winner's first move?"

   Reply: "He will remove three pennies, leaving the ultimate loser confronted with twelve pennies".

TURN	ULTIMATE WINNER'S TURN	ULTIMATE LOSER'S TURN	# PENNIES CONFRONTING HIM	# PENNIES THAT HE WILL CHOOSE
1st	X		15	0
2nd		X	12	1-5
3rd	X		7-11	1-5
4th		X	6	1-5
5th	X		1-5	1-5
6th		X	0	1-5

Therefore, in order to guarantee victory, the above format must be followed.

Problem 10

Label the first losing player P_1, the second P_2, and the third P_3.

GAME	P_1	P_2	P_3
End of game #3	\$8	\$8	\$8
End of game #2	\$4	\$4	\$16
End of game #1	\$2	\$14	\$8
Beginning of game	\$13	\$7	\$4

Therefore, at the beginning of the game the three players had \$13, \$7, and \$4 respectively.

8. Lesson Eight

Time	Topic
	Problem Solving (types #2 and #3)
40 min.	Lesson Plan Outline

The teacher will correct problem eleven (with input from the students), writing out the solution on the blackboard. He will then do problem twelve for the pupils in order to demonstrate the solution strategy for type #3 problems. The solutions for problems eleven and twelve are located in this lesson under the heading: "Answers to Problems For the Eighth Day".

The instructor will then assign problem thirteen to be worked on during the period and to be completed for the next lesson.

ANSWERS TO PROBLEMS FOR THE EIGHTH DAY

Problem 11

The optimal solution is summarized in the chart on the next page.

STATE	CHECKER ARRANGEMENT	ARRANGEMENT NUMBER
Goal State	B B B W W	1
	B W B B W	2
	B B W B W	3
	B W B B W	4
Initial State	B W B W B	5

Therefore, in progressing through the game, the player starts in the initial state (arrangement #5 in the above chart) and makes the necessary moves to arrive at arrangement #4, #3, #2, and finally at #1 (the goal state).

<u>Problem 12</u>

Demonstrate that it is impossible for Bob to place a different number of coins into each of 10 pockets, given that he only has 44 coins.

First pocket gets 0 coins

Second pocket gets 1 coin

Third pocket gets 2 coins

.
.
.

Tenth pocket gets 9 coins

The minimum number of coins = 1+2+3+ ... +9 = 45 coins. Therefore, since Bob only has 44 coins, he cannot put a different number into each pocket.

9. Lesson Nine

Time	Topic
	Problem Solving (type #3)
40 min.	Lesson Plan Outline

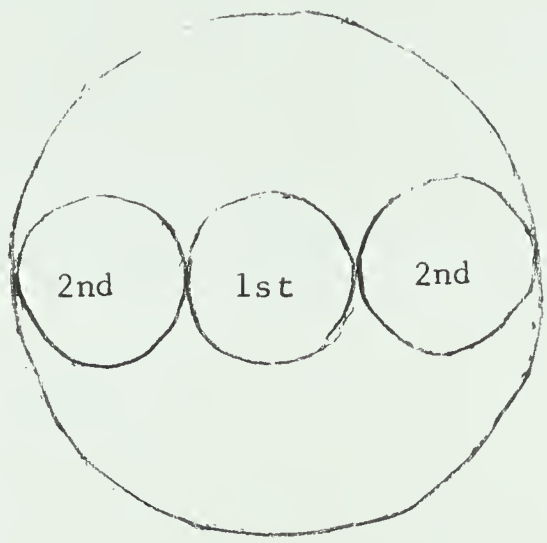
The teacher will correct problem thirteen (with input from the students), writing out the solution on the blackboard. The solution is located in this lesson under the heading: "Answer to Problem For the Ninth Day".

He will then assign problem fourteen to be worked on during the period and to be completed for the next lesson.

ANSWER TO PROBLEM FOR THE NINTH DAY

Problem Thirteen

In solving the problem, the first player initially places a poker chip in the center of the table and thereafter plays chips in a symmetrically opposite position to that played by the second player.



Clearly, if the second player has any place on the table available to place a poker chip, there will still be a symmetrically opposite place on the table for the first player to place a chip, so that the first player must be the last to play a chip on the table, independent of the size of the table.

10. Lesson Ten

Time	Topic
	Problem Solving (types #3 and #4)

40 min.

Lesson Plan Outline

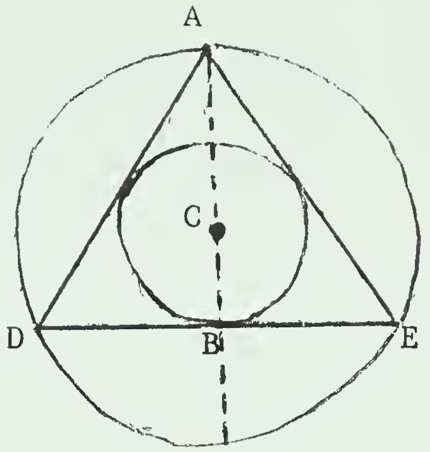
The teacher will correct problem fourteen (with input from the students), writing out the solution on the blackboard. He will then do problem fifteen for the pupils in order to demonstrate the solution strategy for type #4 problems. The solutions for problems fourteen and fifteen are located in this lesson under the heading: "Answers to Problems For the Tenth Day".

The instructor will then assign problem sixteen to be worked on during the period and to be completed for the next lesson.

ANSWERS TO PROBLEMS FOR THE TENTH DAY

Problem 14

Given an equilateral triangle with an inscribed circle (with radius r), a circumscribed circle (with radius R), and its longest altitude (H), is $r+R \leq H$?



$r = H/3$
$R = (2H)/3$

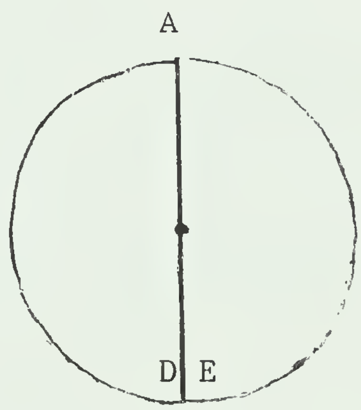
$\overline{AB}$ = median of equilateral triangle
$= H$

Therefore $2(\overline{BC}) = \overline{AC}$

Thus $r = \overline{BC}$
$= H/3$
$R = \overline{AC}$
$= (2H)/3$

In this case, the theorem is true.

A more extended case is that of the isosceles triangle. The form of an isosceles triangle varies with the angle at the vertex and there are two extreme (or limiting) cases; the one in which the angle at the vertex becomes 0 degrees, and the other in which it becomes 180 degrees. In the first extreme case the base of the isosceles triangle vanishes and



$R = H/2$

$r = 0$

For this case, the theorem is true.

In the second limiting case, however, the height vanishes and

$r = 0$

$R = \infty$
(Circumscribed circle must pass through points A, D, and E).

$H = 0$

For this case, the theorem is not true. Therefore, the theorem is false, and the problem is solved.

Problem 15

The instructor will begin be drawing a diagram depicting the original situation.

First:

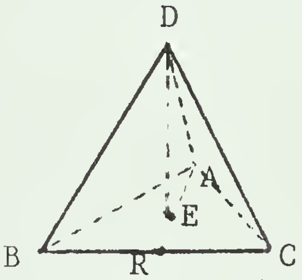
Triangle ADE is a right triangle (because $\overline{DE}$ is the altitude).

Therefore $\overline{AD}^2 = \overline{AE}^2 + \overline{DE}^2$ (Pythagoras' theorem)

or $\overline{DE} = \sqrt{\overline{AD}^2 - \overline{AE}^2}$

but $\overline{AD} = 10$ cm (edge of tetrahedron)

Therefore $\overline{DE} = \sqrt{100 - \overline{AE}^2}$



(tetrahedron)

Second:

Find $\overline{AE}$.

Examine triangle ABC.

Given altitude $\overline{AR}$.

$\overline{ER} = \frac{1}{2}\overline{AE}$

Find $\overline{AR}$.

Examine right triangle ARC.

Pythagoras' theorem states:

$$\overline{AC}^2 = \overline{RC}^2 + \overline{AR}^2$$

$$10^2 = 5^2 + \overline{AR}^2$$

$$\overline{AR} = \sqrt{10^2 - 5^2}$$

$$= 5\sqrt{3} \text{ cm}$$

but $\overline{AR} = \overline{AE} + \overline{ER} = 5\sqrt{3}$ cm

$= \overline{AE} + \frac{1}{2}\overline{AE} = 5\sqrt{3}$ cm

Therefore $(3/2)(\overline{AE}) = 5\sqrt{3}$ cm

$$\overline{AE} = 10\sqrt{3}/3 \text{ cm}$$

<u>Third</u>:

$\overline{DE} = \sqrt{100 - AE^2}$

$= \sqrt{100 - 100(3)/9}$

$= \sqrt{(900-300)/9}$

$= 10\sqrt{6}/3$ cm

$= 8.16$ cm

Therefore, the height of the regular tetrahedron is 8.16 cm.

11. Lesson Eleven

Time	Topic
	Problem Solving (type #4)
40 min.	**Lesson Plan Outline**

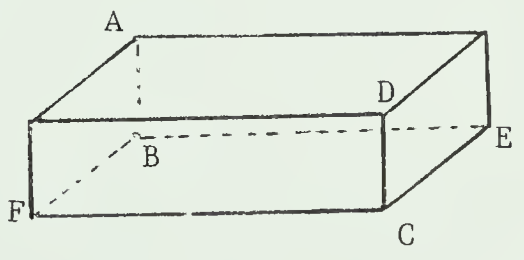
The teacher will correct problem sixteen (with input from the students), writing out the solution on the blackboard. The solution is located in this lesson under the heading: "Answer to Problem For the Eleventh Day".

He will then assign problem seventeen to be worked on during the period and to be completed for the next lesson.

ANSWER TO PROBLEM FOR THE ELEVENTH DAY

Problem 16

Draw a diagram of the rectangular solid, length "a" cm , width "b" cm , and height "c" cm.



The problem is to find the length of $\overline{AC}$ (or $\overline{BD}$).

Pythagoras' theorem states:

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$$

$$\overline{AC} = \sqrt{\overline{AB}^2 + \overline{BC}^2}$$

$$= \sqrt{c^2 + \overline{BC}^2}$$

<u>First</u>:

Take the rectangular base of the solid.

$$\overline{BC}^2 = \overline{BE}^2 + \overline{EC}^2 \quad \text{(Pythagoras' theorem)}$$

$$= a^2 + b^2$$

<u>Second</u>:

Therefore $\overline{AC} = \sqrt{c^2 + \overline{BC}^2}$

$$= \sqrt{c^2 + (a^2+b^2)}$$

Thus the length of the diagonal of the rectangular solid is $\sqrt{a^2 + b^2 + c^2}$ cm.

12. Lesson Twelve

Time	Topic
	Problem Solving (types #4 and #5)

40 min.

Lesson Plan Outline

The teacher will correct problem seventeen (with input from the students), writing out the solution on the blackboard. He will then do problem eighteen for the pupils in order to demonstrate the solution strategy for type #5 problems. The solutions for problems seventeen and eighteen are located in this lesson under the heading: "Answers to Problems For the Twelfth Day".

The instructor will then assign problem nineteen to be worked on during the period and to be completed for the next lesson.

ANSWERS TO PROBLEMS FOR THE TWELFTH DAY

Problem 17

The equation depicting the original situation is given by:

$$4x^4 - 33x^2 = 27 \text{ or}$$

$$4x^4 - 33x^2 - 27 = 0$$

where x is the number of dollars in the father's wallet.

Let $y = x^2$

Then $4y^2 - 33y - 27 = 0$

Solve the new problem.

$4y^2 - 33y - 27 = 0$

$(4y+3)(y-9) = 0$

$y = -3/4, 9$

If $y = -3/4$

$x^2 = -3/4$

x = an inadmissable value

If $y = 9$

$x^2 = 9$

$x = 3, -3$

$x = -3$ is inadmissable

Therefore, there are \$3 in the father's wallet.

Problem 18

Let \$x be the salary per month.

$(.18)(x) = 45$ or $(18)/(100) = (45)/(x)$

$x = 45/(.18)$

$= 250$

Therefore, his salary for a month is \$250.

13. Lesson Thirteen

Time	Topic
	Problem Solving (type #5)
40 min.	Lesson Plan Outline

The teacher will correct problem nineteen (with input from the students), writing out the solution on the blackboard. The solution is located in this lesson under the heading: "Answer to Problem For the Thirteenth Day".

He will then assign problems twenty and twenty-one to be worked on during the period and to be completed for the next lesson.

ANSWER TO PROBLEM FOR THE THIRTEENTH DAY

Problem 19

Let x years be Anne's age now.

Let y years be Jane's age now.

$$x = 7+y \quad \text{(call this equation \#1)}$$

$$x-1 = 2(y-1) \quad \text{(call this equation \#2)}$$

Substitute equation #1 into equation #2.

$$(7+y) - 1 = 2(y-1)$$

$$y+6 = 2y-2$$

$$y = 8 \text{ years.}$$

Solve for x in equation #1.

$$x = 7+8$$

$$= 15 \text{ years.}$$

Check in equation #2.

$$x-1 = 2(y-1)$$

$$15-1 = 2(8-1)$$

$$14 = 14$$

Therefore, Anne is 15 years old and Jnae is 8 years old.

14. Lesson Fourteen

Time	Topic
	Problem Solving (type #5)
40 min.	**Lesson Plan Outline**

The teacher will correct problems twenty and twenty-one (with input from the students), writing out the solutions on the blackboard. The solutions are located in this lesson under the heading: "Answers to Problems For the Fourteenth Day".

He will then assign problems twenty-two and twenty-three to be worked on during the period and to be completed for the next lesson.

ANSWERS TO PROBLEMS FOR THE FOURTEENTH DAY

Problem 20

Let x be the larger number.

Let y be the smaller number.

$$x-y = 10 \quad \text{(call this equation \#1)}$$

$$-2x+5y = 22 \quad \text{(call this equation \#2)}$$

Multiply equation #1 by 2.

$$2x-2y = 20 \quad \text{(call this equation \#3)}$$

Add equation #2 and equation #3.

$$3y = 42$$

$$y = 14$$

Solve for x in equation #1.

$$x-14 = 10$$

$$x = 24$$

Check in equation #2.

$$-2(24)+5(14) = 22$$

$$22 = 22$$

Therefore, the numbers are 14 and 24.

Problem 21

Let x kilograms be amount candy worth \$1.50/kg.

and \$1.5x be the value of candy worth \$1.50/kg.

Let y kilograms be amount candy worth \$2.70/kg.

and \$2.7x be the value of candy worth \$2.70/kg.

$$x+y = 225 \quad \text{(call this equation \#1)}$$

$$1.5x+2.7y = (225)(2.28)$$

$$= 513 \quad \text{(call this equation \#2)}$$

Multiply equation #2 by 10.

$$15x+27y = 5130 \quad \text{(call this equation \#3)}$$

Multiply equation #1 by 15.

$$15x+15y = 3375 \quad \text{(call this equation \#4)}$$

Subtract equation #4 from equation #3.

$$12y = 1755$$

$$y = 146.25 \text{ kg.}$$

Solve for x in equation #1.

$$x+146.25 = 225$$

$$x = 78.75 \text{ kg.}$$

Check in equation #2.

$$(1.5)(78.75)+(2.7)(146.25) = 513$$

$$513 = 513$$

Therefore, there are 78.75 kilograms worth $1.50/kg. and 146.25 kilograms worth $2.70/kg.

15. Lesson Fifteen

Time | Topic

Problem Solving (type #5)

40 min.

Lesson Plan Outline

The teacher will correct problems twenty-two and twenty-three (with input from the students), writing out the solutions on the blackboard. The solutions are located in this lesson under the heading: "Answers to Problems For the Fifteenth Day".

He will then assign problem twenty-four to be worked on during the period and to be completed for the next lesson.

ANSWERS TO PROBLEMS FOR THE FIFTEENTH DAY

Problem 22

Let x be the number of sports cars in John's dream.

Let y be the number of pilots in John's dream.

$x-y = 7$ (call this equation #1)

$x-(y/2) = 29$ (call this equation #2)

Subtract equation #1 from equation #2.

$y/2 = 22$

$y = 44$

Solve for x in equation #1.

$$x-y = 7$$

$$x-44 = 7$$

$$x = 51$$

Check in equation #2.

$$x-(y/2) = 29$$

$$51-(44/2) = 29$$

$$29 = 29$$

Therefore, there were 51 sports cars in John's dream.

Problem 23

Let \$x be the amount invested at 5% per annum.

Let \$y be the amount invested at 6% per annum.

$$I_1 + I_2 = \$200$$

$$\begin{aligned} I_1 &= prt \\ &= (x)(.05)(1) \\ &= .05x \end{aligned}$$

$$\begin{aligned} I_2 &= prt \\ &= (y)(.06)(1) \\ &= .06y \end{aligned}$$

Therefore, $(.05x) + (.06y) = 200$

or $5x + 6y = 20000$ (call this equation #1)

And $y = 1500 + x$ (call this equation #2)

Substitute equation #2 into equation #1.

$$5x + 6(1500+x) = 20000$$

$$11x + 9000 = 20000$$

$$x = 1000$$

Solve for y in equation #1.

$$y = 1500 + x$$

$$y = 1500 + 1000$$

$$y = 2500$$

Check both solutions in equation #2.

$$5x + 6y = 20000$$

$$5(1000) + 6(2500) = 20000$$

$$20000 = 20000$$

Therefore, the amount invested at 5% was $1000 and the amount invested at 6% was $2500.

16. Lesson Sixteen

Time	Topic
	Problem Solving (type #5)

40 min.

Lesson Plan Outline

The teacher will correct problem twenty-four (with input from the students), writing out the solution on the blackboard. The solution is located in this lesson under the heading: "Answer to Problem For the Sixteenth Day".

In the time remaining in the period, he will discuss the solutions to any of the twenty-four problems in which the pupils are still having difficulties.

ANSWER TO PROBLEM FOR THE SIXTEENTH DAY

Problem 24

Let x be the number of hens.

Let y be the number of rabbits.

$2x+4y = 140$ (call this equation #1)

$x+y = 50$ (call this equation #2)

Divide equation #1 by 2.

$x+2y = 70$ (call this equation #3)

Subtract equation #2 from equation #3.

$$y = 20$$

Solve for x in equation #1.

$$2x+4y = 140$$

$$2x+4(20) = 140$$

$$2x = 60$$

$$x = 30$$

Check both solutions in equation #2.

$$x+y = 50$$

$$30+20 = 50$$

$$50 = 50$$

Therefore, there are 30 hens and 20 rabbits.

17. Lesson Seventeen

Time	Topic
	Heuristic Problem-Solving Test
75 min.	Administration of the Test

The instructor will hand out the tests and in doing so, he will tell the students to fill in their names on the first page of the test. The test is contained in this thesis in Appendix C, and the scoring procedures are contained in Appendix G (lesson seventeen).

18. <u>Lesson Eighteen</u>

Time	Topic
	Student Opinion Survey
20 min.	<u>Administration of the Survey</u> The instructor will hand out the tests and in doing so, he will tell the students <u>not</u> to write their names on survey. The survey is contained in this thesis in Appendix E.

19. Lesson Nineteen

Time	Topic
	Mathematics 20 Problem-Solving Test (post-test)
70 min.	Administration of the Test

The instructor will hand out the tests and in doing so, he will tell the students to fill in their names on the first page of the test. The test is contained in this thesis in Appendix A, and the scoring procedures are contained in Appendix G (lesson one).

APPENDIX I

EXPERIMENTAL GROUP'S QUALITATIVE RESPONSES TO THE STUDENT OPINION SURVEY

EXPERIMENTAL GROUP'S QUALITATIVE RESPONSES TO THE STUDENT OPINION SURVEY

1. Question One

"I feel it was helpful because going through a problem step by step really helps you get to know how to do it."

"I helped us organize our thoughts and procedures in problem solving."

"The procedure was helpful because it broke the problem down and you could look at what info you had to come up with a method to go about solving it."

"I found that it was helpful because it organized my thoughts and helped me figure out how to solve the problem."

"Yes, I can now solve some problems that I couldn't before."

"No, it wasn't because I want to know mentally how to get the answer, not to follow a written procedure."

"No, it is not necessary to follow the procedure. The statistics might show my problem-solving ability has improved but this is only because of practice."

"Yes. You know what you were looking for and with what."

"Yes. It laid the problem out and it did make it easier to solve."

"Sure it was helpful cause every time you forgot what you just read, you could look at your UNKNOWN and GIVEN."

"No, it was a kind of waste of time because you could remember such things."

"Yes, to understand the methods of solution."

"Yes, they were helpful. They gave me a method or procedure of how to go about solving problems."

"Yes, because it had stated clear and more easily understood."

"Yes."

"Yes, it helped to get yourself organized from the start."

"Yes it was, for it sorted the problem out better, but very tedious. You have to work for 5 min. before you reach the equation."

"This procedure was helpful; at first it seemed hard and dragged out but it was an aid."

"Yes, I never did really know how to solve problems properly."

"Yes, it was easier to analyze problems and see what you were asked and given, making the problem easier to solve."

"Yes it did because I knew what I needed and was looking for."

"Yes, you do not have to look back to the problem to remember what you are solving for."

"Yes, it put all the part of the problem in sections so it was easier to understand."

2. Question Four

"Yes, I feel I can solve more problems now than before."

"Sort of."

"Yes, because now I am able to look at a problem and evaluate it without getting too frustrated or discouraged."

"Yes I do believe I have improved because now I have some idea of how to go about solving a problem; it is no longer hit and miss."

"Yes. As much as I hate to admit it, I do. I can better understand the problem."

"No. I'm still as dumb as before."

"Yes, I have had more practice in problem solving."

"Not much better. I didn't understand much of it."

"No. The course was very good but I haven't improved very much at all."

"Yes, a bit better. Probably because we were being taught."

"Yes, because now I have a procedure to follow."

"Yes, because all the procedure we had learnt was understood but sometime find works difficult."

"Yes."

"Yes, because it takes me less time to work out the answer to the problem and now I know of some methods and how to go about doing it."

"Yes, I couldn't do some of the problems or the pretest."

"Yes, now I know how to solve a problem with every type of solution."

"Yes; can't think of a good explanatory reason, though."

"I feel I have improved slightly but I also feel that my eyes have been opened to a lot I didn't know."

"Yes, I know the right procedures."

"Yes; I feel I can solve the type of problem you gave, but I might not be able to apply this to problems that aren't so similar in detail."

"Yes, they're coming easier."

"Yes."

"Yes, I have now worked with a lot of different problems and can figure out similar ones easier."

3. <u>Question Five</u>

"Probably, if we have any problem like this."

"Yes, because now I know that no matter what problem I know that I know a method of solution to solve it and it isn't hopeless."

"Yes, it has given methods that are easier than using straight algebra to solve questions."

"Yes. I don't know. I guess I can sort the problem out more clearly now."

"Maybe it will, maybe it won't."

"Yes, this unit has shown me new methods of solution and has given me practice in decomposing and recombining."

"No. It's too hard guessing what the characteristics are."

"I would apply what I learnt and see if it helps. I think it will."

"Yes, but if it doesn't it's always good to have."

"Yes, because now I have a procedure to follow."

"Yes. By the procedures or steps that we have learnt such as Unknown, Given, Related problem, Characteristics, method of soln, etc.."

"Yes."

"Yes, I think they will help me solve new problems in Math 20 in that there is now a set of steps for me to follow."

"Yes."

"Yes, because you know what to do."

"Yes, I know that if I get stuck, I can refer to what I have learned and apply it."

"I guess it should, for I will be more aware of what I am looking for in the problem."

"Yes and no. Some of the methods I am clear on whereas some I'm not."

"Yes; I know it will help some in analyzing problems, but could find some difficulty since they could be quite different."

"Yes, because we have an organized way of approaching the problem."

"Yes, we can relate back to them."

"Yes."

"Yes, it will give me an idea as to how to solve new problems."

4. Question Six

"Very good course."

"The unit was easy to follow and understand."

"At first I didn't like the unit but towards the end I did start to enjoy it."

"I didn't like all the tests! And I didn't like being timed on them. I think it made me rush too much. When you were really into a problem, all of a sudden it was time to go on to the next one, and my concentration was broken."

"Try not to teach so specificly (traditional); ease up a bit, change the way of teaching. OK."

"I thought this was a waste. I think it helped a bit but not as much as I had hoped."

"The course was very good. It was presented very well."

"It was a good course but was too fast."

"It was a fairly good course."

"Sometimes, too difficult to understand the question."

"I enjoyed you as a teacher."

"Your blackboard erasing technique is very original and humorous."

"I think that the course was all in all a good one, but more time should be spent on the specialization and analogy and less on the generalization and decomposing and recombining."

"Could have spent more time on showing us how to use each of the methods."

"Good course, but maybe should show additional examples, ones differing from each other."

"Should have gone into them deeper."

"Very helpful and useful course; interesting."

APPENDIX J

CONTROL GROUP'S QUALITATIVE RESPONSES

TO THE STUDENT OPINION SURVEY

CONTOL GROUP'S QUALITATIVE RESPONSES TO THE STUDENT OPINION SURVEY

1. Question One

"Yes, because you would know what format to start with in the problem."

"Yes, you learn it if you have to think it out but are helped along with hints, etc.."

"Yes, it gave you something to work with (given)."

"Yes."

"Yes and no. Sometimes I was unable to follow the process the problem was solved by."

"Yes, because it showed us that there was more than one way to solve problems."

"Yes—it gave different ideas for solving different basic types of problems."

"Yes, because you can use one of the types for every procedure presented. Each type has something different about it."

"Yes, because you have to know what you are trying to find and what you know already in order to solve problems."

"Yes, it is because I have more information about the problems."

"No, I think an exact formula would be the most efficient."

"Yes."

"No. I think most of the problems were stupid questions anyway."

"Yes. It gives a starting point to the problem."

"Just a little. It slightly changed my perspective towards the order of treating a problem."

"Yes, I cam up with the answers quicker and it was easier to transfer thoughts to the paper with the procedure."

"I found it easier to do some problems than others."

"No, because I'm too used to problems which can be formed into equations."

"No, I still find it extremely difficult to solve problems."

"I felt I improved slightly but not as much as I could have, because I have trouble comprehending problems."

2. Question Four

"Now, because charts showed how to solve questions, I didn't know before."

"Yes, because I now know how to approach a problem better."

"No, not enough."

"Yes, I managed to learn a little more technique."

"Yes, I have learned the method."

"Yes, because I have been exposed to new and different methods of solution and I was given interesting problems showing me that problem solving isn't always boring."

"For certain problems like type #1, with the chart. Other than that, not really."

"Yes, because I can understand the problems better and therefore be able to solve them better."

"Yes, because I have learned many ways to solve a problem."

"I didn't feel better than before because I don't like you cause."

"Yes, because the course gave practice and showed different ways of solving problems."

"No. Have not had enough experience at it."

"Yes. Before I never really had any standard procedures to follow."

"No. I have yet to come across a problem like them outside a math class."

"A little bit better. I learned to look at a problem in more than one way."

"Yes, the problems presented are more practical."

"Not likely because there was not enough instructing time spent during seminars."

"Yes, I can put my thoughts down on paper easier now than before. Before I could answer the questions in my head but not be able to put them down on paper."

"No; I may have learned a little, but still not enough to solve problems."

"No, because not enough practice was given in each of the different types of problems. I didn't catch on."

"No, I'm no better than before and my test proved it. I find it difficult to know what is needed from the problem to solve it. I can't pick what special situation is needed."

"No and yes; because some problems are easier and some are harder."

3. Question Five

"Depends on what the topics are in Math 20."

"Yes, because the things we learned in the unit can be applied to the other new problems."

"Yes, I've got the experience."

"Yes."

"Yes, because I have learned the method."

"Yes, because I can solve by one method and check by another."

"It could be revealing the different basic types of problems and how to go about solving them."

"Yes, because I can relate the problems to one of the five types and be able to find the solution easier."

"Yes, it helps me sort out the unknown and the known."

"I don't think so."

"Yes, cause I know more ways to figure out problems."

"Yes, it gives me the basics I need to know."

"Yes, the procedure is clearer."

"Most or all of the problems don't make a damn bit of sense to me."

"Yes, it trains the mind to look at a problem from different angles."

"Yes."

"No comment."

"Yes."

"Yes and no. Some problems were hard and others weren't. It depends on the type of problem."

"No, because like I said, I didn't catch on. I didn't learn very much."

"No."

"I suppose they will because I'll know how to set up problems."

4. Question Six

"I thought it was a fun way of approaching those problems."

"You never see any problems like those outside a math room."

"I think the test should have had more questions of each type so you can pick one of each instead of being forced to pick one specific one."

"I really enjoyed this unit because it refreshed my memory on different things I had learned and now maybe I won't forget them whereas the curriculum for grade 12 and the rest of 11 may not include any of this."

"I think it was bad having to cut down our regular math time and to make the problem solving marks count for the Math 20 mark."

"This was helpful, but the tests were too difficult."

"Problem solving needs so much practice."

"No good at all in Math 20."

"I liked the unit."

"A big unit like this needs much more time."

"I think programs such as this should be started in grade 9 to save the students from 2 years of unsolvable problems."

"Why do they have those kinds of problems any way. I don't think I will ever see one of those problems outside a math class. Utter nonsense."

"Wish to have more chance to do this type of problem solving."

"Was hard to do both logs and problem solving at the same time."

"If more practice would have been given in different types of problem solving, I would have done a lot better on the tests and in the future."

"It took up a lot of our time."

"I feel that special solutions should have been more clear."